Comparison of Estimation Formulae for the Length and Order of Polynomial-based Filter

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***Abstract*— The two basic design parameters for polynomial-based digital interpolation filters are number of polynomial-segments defining the finite length of impulse response, and order of polynomials in each polynomial segment. The complexity of the implementation structure and frequency domain performance depend on these two parameters. This contribution compares two types of estimation formulae for length and polynomial order of polynomial-based filters for various types of requirements including attenuation in stopband, width of transitions band, deviation in passband, weighting in passband/stopband.**

***Index Terms*— Digital filters; Filter design; Polynomial-based filters; Farrow structure.**

I. Introduction

THE digital polynomial-based interpolation filters are used in signal processing applications whenever it is required to calculate signal samples at arbitrary positions between existing samples. The impulse response of digital polynomial-based interpolation filters is derived by using piecewise polynomial analogue model [1]. An underlying continuous-time impulse response $h_a(t)$ has the following properties [1]; First, $h_a(t)$ is nonzero only in a finite interval $0 \leq t \leq NT$ with N being an integer which determines the length of the filter. Second, in each subinterval $nT \leq t < (n+1)T$, for $n= 0,1\ldots, N-1$, $h_a(t)$ is expressible as a polynomial of t of a given order M. Third, $h_a(t)$ is symmetric with respect to its middle point t = $NT/2$, to guarantee phase linearity of the resulting overall system. The advantage of the above impulse response is in the fact that the actual implementation can be efficiently performed by using the Farrow structure [2] or its modifications such as transposed Farrow structure, prolonged Farrow structure etc [1], [3]. The number of multipliers in any modification of Farrow structure is directly proportional to polynomial order M and number of polynomial segments N.

The polynomial-based interpolation filters can be designed using time domain or frequency domain design methods [1]. The time-domain design methods are based on Lagrange and B-spline interpolations, where the approximating polynomial is fitted to the discrete-time samples. In the frequency domain design methods, the coefficients of the polynomials, i.e. coefficient of the Farrow structure, are optimized directly in the frequency domain [1], [3]. In all these design methods, there are two basic parameters that control performance of the filter and its complexity. These two basic parameters are the number of polynomial segments N and polynomial order M.

As stated above, polynomial order M and filter length N are directly proportional to the number of multipliers in Farrow based structure [1], [2]. Thus, the system complexity measured in number of operations is directly related to these two parameters. Furthermore, the system performance in the frequency domain measured by stopband attenuation and passband ripple is also related to N and M. Therefore, it is very important to estimate polynomial order M and filter length N according to given system parameters.

Various order estimation formulae exist for FIR filters, for example Kaiser order estimation [4]. In the actual digital implementation, polynomial-based filters can be modeled as FIR filters [1], thus we can use similar methodology for analysis. The estimation formula for N, which can be found in [5] is good starting point for filter length N. In [6], written by the second author Babic et al., the estimation formulae for both N and M have been proposed. The conference paper [6], has been main motivation for research presented in [7] and [8]. However, the formulae presented in [6] are obtained using trial and error method, and they have some conditional restrictions. Namely, the estimation formulae of [6] cannot be used in the case when filter with narrow transition band is designed. In [7], the estimation formulae for the prolonged Farrow structure has been proposed. In [8], we have proposed more general and accurate formulae for estimation of polynomial order M and filter length N based on the system requirements including attenuation in stopband, width of transitions band, deviation in passband, weighting in passband/stopband. The formulae presented in [6] and [8] can save time for the filter designers, because they get suitable starting values for N and M for the given set of requirements. They can serve to estimate the filter complexity for given set of system requirements.

In this paper, we compare the estimation formulae for the length N and polynomial order M given in [6] and [8]. The experimental results which are used for estimation are illustrated graphically together with estimation formulae. The formulae are also tested by means of design examples.

II. Review of Estimation Formulae for the Length N and Order M of Polynomial-based Interpolation Filter

As mentioned above, the cost of realization, i.e. the number of multipliers, of a polynomial-based interpolation filter can be estimated by introducing the required values for N and M into design method [1], [2]. It would be very beneficial to

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estimate N and M by only using the given specifications of the filter in the frequency domain. Similar order estimation formulae exist for FIR filters, for example Kaiser order estimation [4]. The formulae of [6] and [8] are obtained experimentally. Many filters were designed, by using different system specifications, in order to adapt the Kaiser formula to the polynomial-based case. The polynomial based filters are designed using minimax optimization routine explained in [1] for the following design parameters. The passband edge is varied from Δf_p to $F/2$-Δf_p with step equal to Δf_p =0.05 normalized to F, where F is output/input sampling rate in decimation/interpolation case. The stopband edge f_s is determined according to case A specification defined in [1], thus f_s=0.5 normalized to F. The weighting function $W(f)$, which is used to differentiate precision of design in passband and stopband, is also varied from $W(f)=[W_p\ W_s]=[1\ 0.1]$ to $W(f)=[1\ 1000]$, where W_p and W_s are weights in passband and stopband region respectively. Finally, we used different values for the number of polynomial segments N ranging from N=2 to N=24 with step two, taking only even values of N. The polynomial order M takes values from M=0 to M=7 with unity step. In this way, in our experiment we use 9 different values of f_p, five values of $W(f)$, 12 values of N, and 8 values of M. All together, we have designed 9·5·12·8=4320 filters. For each set of requirements, we have determined obtained performance in terms of passband ripple δ_p, and stopband ripple δ_s.

The estimation formula for N, which has been presented in [6], is

$$N_e = 2\left\lceil \frac{-20\log_{10}(\sqrt{\delta_p \delta_s}) - 8.4}{30.4(f_s - f_p)/F} \right\rceil \quad (1)$$

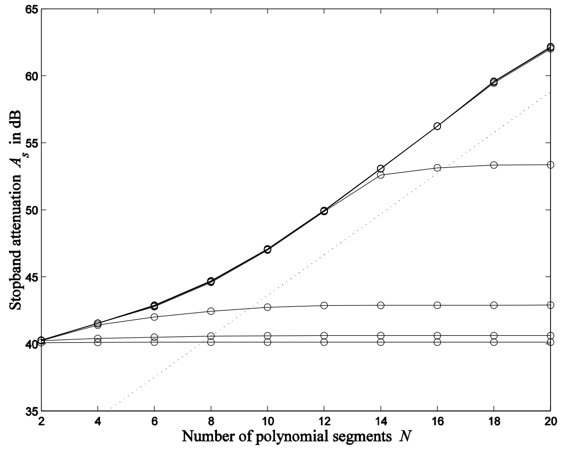
where δ_p and δ_s are the maximum deviations of the amplitude response from unity for $f\in[0,f_p]$ and the maximum deviation from zero for $f\in\Phi_s$, respectively. Here, $\lceil x \rceil$ stands for the smallest integer which is larger or equal to x. It has been observed that in most cases the above estimation formula is rather accurate with only a 2% error. The formula above is valid for all three types of requirements, i.e., A, B, and C, as given in [1]. However, if the transition band is narrow, i.e., in the case when $(f_s$-$f_p)/F\leq 0.1$, the required value of N should be increased by 2. Further, in the case of very narrow transition band ($(f_s$-$f_p)/F \leq 0.05$) the formula (1) cannot be used. The estimation formula for the number N of polynomial segments can be expressed in a different form:

$$N_e = 2\left\lceil \frac{A_s - 10\log_{10}(W) - 8.4}{30.4(f_s - f_p)/F} \right\rceil \quad (2)$$

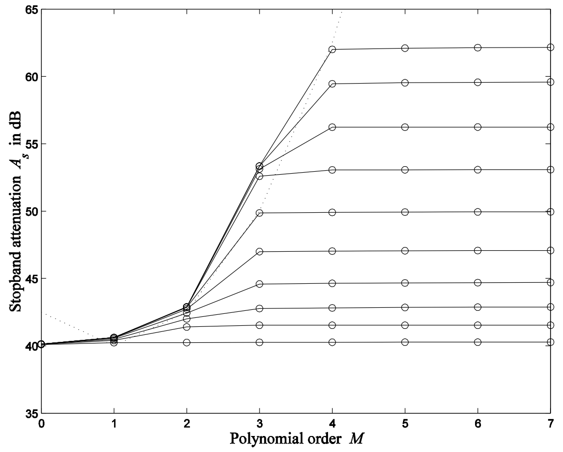
where A_s=-20log$_{10}(\delta_s)$ is the required attenuation in stopband, and $W=\delta_p/\delta_s$ represents weighting between required tolerances in passband and stopband.

The polynomial order M to meet the specifications can be estimated using formula of [6]

$$M_e = \left\lceil \sqrt{\frac{A_s - 20\cdot\log_{10}(W)}{2.5}} + \log_{10}(W) \right\rceil. \quad (3)$$



(a)



(b)

Fig. 1. *Case A specifications*: The passband and stopband edges are at f_p=0.4F and at f_s=0.5F, and stopband weighting W=100. (a) The curves are shown for M equals 0 to 7. Dashed line is plot obtained from the estimation formula (2) for N . (b) The curves are shown for N equals 2 to 20. Dashed line is plot obtained from the estimation formula (3) for M.

It has been observed that if transition band is relatively large to the sampling frequency, that is when $(f_s$-$f_p)/F \geq 0.5$, the required value of polynomial order M is lowered by one. The estimation formula cannot be used when the transition band is very small, i.e., in the case when $(f_s$-$f_p)/F<0.1$. However, even in this border situation required value of M is always smaller than M_e given by (3). Thus, the estimation formula (3) for the polynomial order M can be used to estimate the upper border for M for all types of requirements.

In [8], more precise formulae are presented. The following estimation formula has been proposed for the number of polynomial segments

$$N = 2\cdot\left\lceil 0.5\cdot\left(\left(\frac{-20\log_{10}(\delta_s)}{28\cdot(f_s - f_p)}\right) + \left(\frac{0.45\cdot w}{(f_s - f_p)}\right) - 0.5\cdot w\right)\right\rceil. \quad (4)$$

where δ_p and δ_s are the maximum deviations of the amplitude response from unity for $f\in[0, f_p]$ and the maximum deviation from zero in stopbands, respectively, and w=1-log$_{10}(W)$. Similarly as in (1), here, $\lceil x \rceil$ stands for the smallest integer

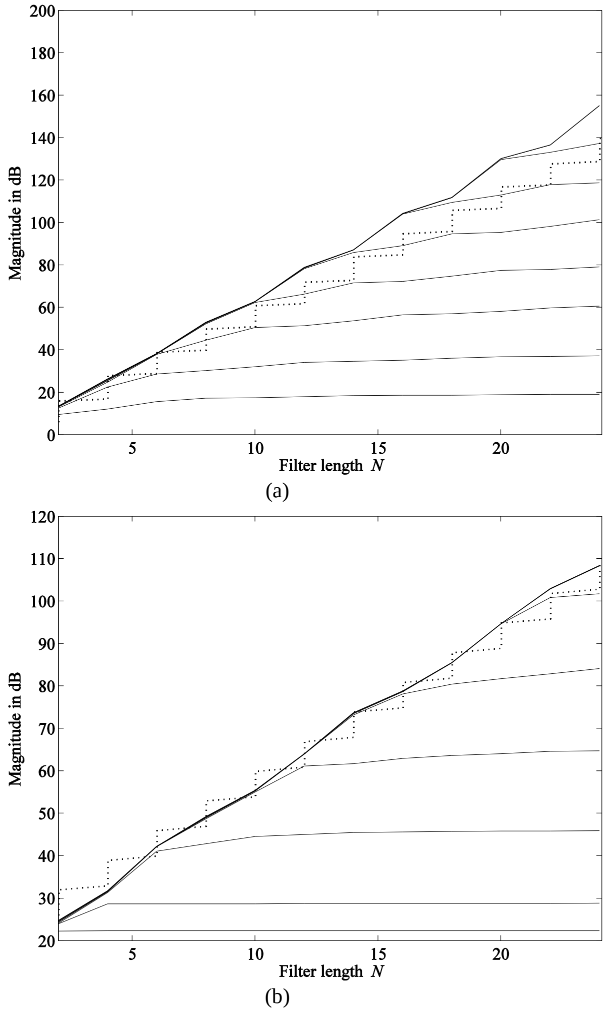


Fig.2 *Case A specifications*: The curves are shown for M equals 0 to 7. Dashed line is obtained from the estimation formula for N shown in (4). The stopband edge is at f_s=0.5 normalized to F, passband edge and stopband weighting are at: (a) f_p=0.15F and W=0.1; (b) f_p=0.3F and W=10.

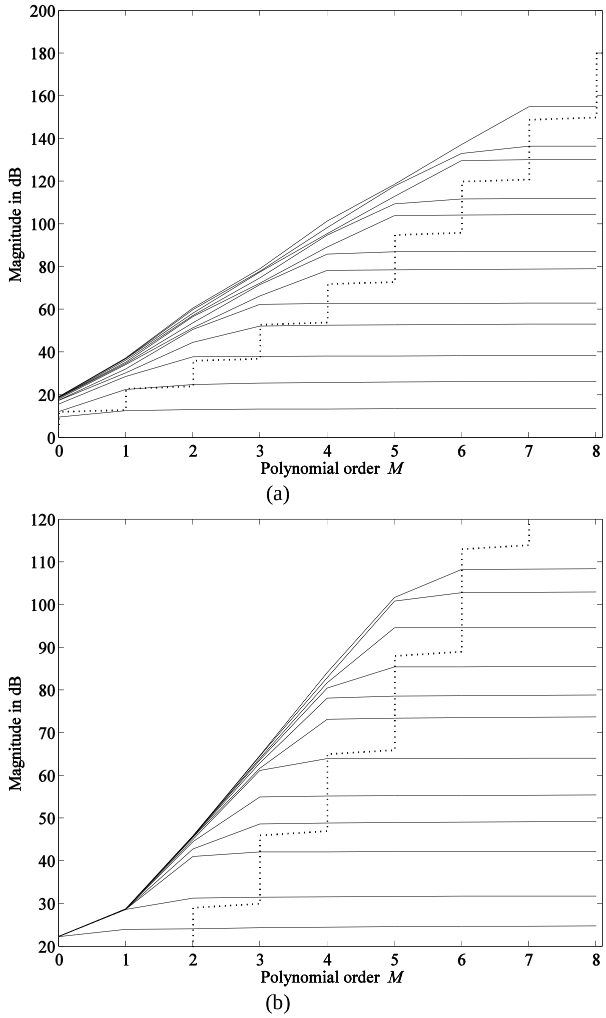


Fig.3 *Case A specifications*: The curves are shown for N equals 2 to 24. Dashed line is obtained from the estimation formula for M shown in (5). The stopband edge is at f_s=0.5, passband edge and stopband weighting are at (a) f_p=0.15F and W=0.1; (b) f_p=0.3F and W=10.

which is larger or equal to x. It has been observed that in the most cases the above estimation formula is rather accurate. However, if the transition band is narrow, i.e., in the case when $(f_s-f_p)/F \leq 0.1$, the required value of N should be increased by 2.

The estimation formula for polynomial order M in [8] has been expressed as:

$$M = \left\lceil \sqrt{\frac{A_s + (10\log_{10}(W) - 40)\cdot \Delta f + 10}{1.5} + \frac{0.13}{\Delta f}} \right\rceil - 3 \qquad (5)$$

where $\Delta f = (f_s - f_p)$. It has been observed that if transition band is relatively large to the sampling frequency, that is when $(f_s-f_p)/F \geq 0.5$, the required value of polynomial order M can be lowered by one. The estimation formula cannot be used when the transition band is very small, i.e., in the case when $(f_s-f_p)/F < 0.1$.

III. DESIGN EXAMPLES

This part gives several examples to illustrate the performance of the presented formulae.

To illustrate this, the following specifications are considered. For the formulae presented in [6], *Case* A *specifications* are considered with passband and stopband edges at f_p=0.4F and at f_s=0.5F. Several filters have been designed in minimax sense with the passband weighting equal to unity and stopband weightings of W=100. The degree of the polynomial in each subinterval M varies from 0 to 7. The number of intervals N varies from 2 to 20. Recall that N is an even integer. Figure 1 give the results for Case A and formulae of [6]. It can be observed that the estimation formulae (2) and (3) are relatively good, as they estimate the border performance for the given set of requirements. Dashed lines in Figs 1 are obtained by using formula (2) in (a) plot and (3) in (b) plot.

For the purpose of illustration of formulae presented in [8], we use the following specifications: *Case A specifications* with stopband edge at f_s=0.5, passband edge and stopband weighting are at f_p=0.15F and W=0.1, in the first example and f_p=0.3F and W=10, in the second example. The curves for both examples are shown in Fig. 2 (a) and (b) respectfully, for N equals 2 to 24. Dashed lines in Fig 2 is obtained from the estimation formula for N shown in (4). A dashed line in Fig.

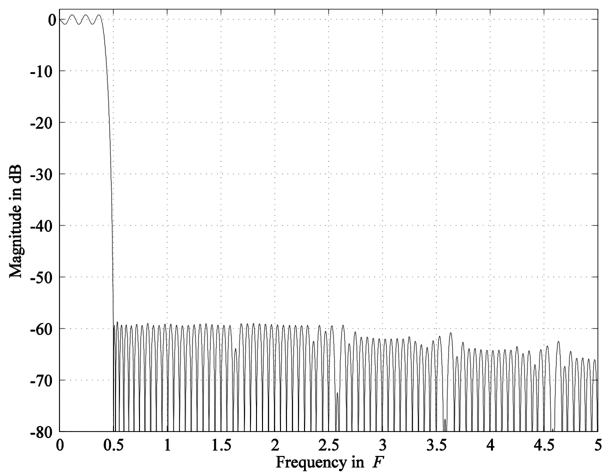


Fig 4. *Design Example*: The frequency domain performance of filter whose parameters are estimated using the presented formulae. The filter specifications are: stopband attenuation A_s=60dB, passband weighting equal to unity and stopband weighting is W=100, passband edge f_p=0.4F. Case A filter of length N=18, and M=4 with achieved A_s=59.5dB and δ_p=0.1065.

3 is a plot obtained from the estimation formula for M shown in (5). One can see that the dashed line represents the border values of M when performance saturates as explained above.

In last example, we can estimate performance of the proposed estimation formulae. The following system requirements are considered: passband and stopband edges are at f_p=0.4F and at f_s=0.5F, the filter is to be designed in minimax sense with stopband attenuation A_s=60dB, passband weighting equal to unity and stopband weightings of W=100.

In order to design filter according to specifications given above, first, we use (2) and (4) to estimate the number of intervals for the Case A specifications. Both formulae give the same result N=18. The degree of the polynomial in each subinterval M=4 is estimated using (3) and (5). We design filter using obtained N and M with earlier defined system requirements. Figure 4 gives the frequency response for Case A filter with achieved stopband attenuation A_s=59.5dB and passband ripple δ_p=0.1065 in linear scale. It can be observed that both estimation formulae are relatively good, as they estimate the border performance for the given set of requirements.

IV. Conclusion

In this paper, we compare the estimation formulae for the number of polynomial segments N and the polynomial order M of polynomial-based filter which are given in [6] and [8]. It has been shown that both sets of the estimations formulae give the satisfactory performance of the filter for the given set of specifications. However, the estimation formulae given in [8] have slightly better performance, especially if system requirements are in border areas. Formulae for N and M can be used to estimate the starting value of these two parameters in minimax optimization used to design the Farrow structure. This is a significant information which can save time for filter designers. Furthermore, the formulae for N and M can be used to estimate implementation costs of the Farrow based filters for the given set of requirements. The formulae can also be used to estimate implementation costs of multistage sampling rate converters containing Farrow filter. Estimation formulae for Case B and Case C requirements with minimax optimization, as well as, least-mean-square optimization are part of the future work.

Acknowledgment

This work is supported by the Ministry of Education, Science and Technological Development of Republic of Serbia under technology development project: TR32028 – "Advanced Techniques for Efficient Use of Spectrum in Wireless Systems".

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