

Analysis of Lossless Quartz Crystals Filters using SchematicSolver and Wolfram Language

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Abstract—SchematicSolver application and Wolfram language are used for the analysis and design of filters that consist of LC sections and quartz crystals. The filter is modelled by LC equivalent sections and represented as rational function. In this paper we present a template notebook for modeling, analyzing, and designing of known filter structure. From the schematic model, the filter transfer function can be derived as closed-form expression so that other analyses are possible in order to satisfy desired filter specifications and responses.

Index Terms—Wolfram language; computer algebra system; Crystal filters, LC filter, Quartz filter.

I. INTRODUCTION

DESIGN and analysis of lossless LC filters are exhaustively described in many books and papers [1]. This approach is suitable for computer analysis because all procedures are well defined. The traditional methodology and developed procedures can be used to design other types of filters. The main disadvantage of manual derivation of responses is usually convenient only for low-order circuits. For the higher-order circuits and complex models is performed using specific software tools and appropriate applications.

The design and analysis of filters with small number of elements can be also time-consuming in the case when each element is replaced with more complex circuit model. Although numeric software can be used for completing the most frequently used task, symbolic approach shows benefits in optimization.

Application of conventional software for the design and analysis of complex filter structure is subject to errors due to the limited accuracy of numerical methods. Analysis of the impact of changes in the value of certain parameters can be done by inverse engineering, but they often insert some errors due to the limited accuracy. In existing software solutions, only approximate and empirical methods were used.

The problem can be solved exactly using computer algebraic methods so that all filter parameters can be describe using symbols [2]. Symbols can be substitute by numeric values only when it is necessary, for example for plotting the graphics of some filter functions. In the case when the

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complexity of the functions is extremely large, certain parameters can be represented as a ratio of integers, and thus retains the accuracy of the final results.

The fact that all parameters are symbols (including complex frequency) provides the possibility to derive the frequency of extremes, to approximate function in a certain range of interest, to replace each element with more complex electrical model, or to compute the sensitivity function to element values. This way the analysis of the impact of certain parameters is facilitated and optimal value of some parameters can be computed under some constrains. As an example, the resonant frequency of a complex filter can be expressed in terms of some filter elements.

II. SCHEMATIC SOLVER

Software SchematicSolver [3] is distributed in the form of notebook, with all descriptions and programs. The first part of the notebook consists of interactive drawing schemes where crystal units are described as impedance. From the schematic, all the essential characteristics can be automatically derived. Verification can be done for cases that have been published in technical papers and results obtained using numeric software.

In some papers it was presented how the knowledge for drawing and analysis can be generated in the notebook. In this paper we are using GUI (Graphical User Interface) [4] of the software SchematicSolver for drawing schematic of some parts of larger filter and the overall filter. The same description of the filter schematic is used for converting symbolic description of the schematic into system of equation that is used for solving the system, such as to find the transfer function.

A. Schematic of impedances and crystal units

The description of several quartz crystals filters can be found in papers [5-7]. The crystal unit can be represented by simple model that is illustrated in Fig. 1. The description of the unit is a list of elements:

```
ZcuSch = {
{"Capacitor", {{2,7},{2, 10}}, cm, "Cm"},
{"Capacitor", {{4, 10}, {4, 1}}, c0, "C0"},
{"GroundE", {3, 1}, 0, " ", TextOffset -> {0,0}},
{"Inductor", {{2, 4}, {2, 7}}, lm, "Lm"},
{"Input", {3,10},Vcu, " ", TextOffset -> {0,-1}},
{"Line", {{2, 10}, {3, 10}}},
{"Line", {{2, 1}, {3, 1}}},
{"Line", {{3, 1}, {4, 1}}},
{"Line", {{3, 10}, {4, 10}}},
{"Polyline", {{0,-1},{0,11},{6,11},{6,-1},{0,-1}}},
{"Resistor", {{2, 1}, {2, 4}}, rm, "Rm"}
};
```

The equivalent impedance of this unit is a ratio of the voltage across the unit and the currents through the unit:

$$Z_{cu} = V_{cu} / (V_{cu} / ((1 / (s C_m)) + R_m + s L_m) + V_{cu} / (1 / (s C_0)))$$

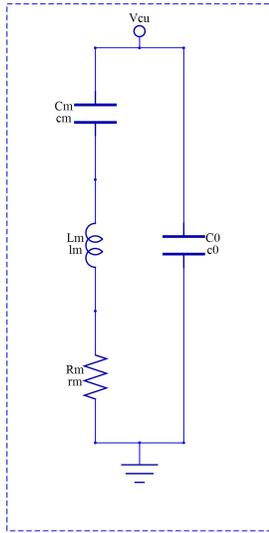


Fig. 1. The model of the crystal unit.

Suppose that the applied voltage is $V_{cu} = V_x$. The resultant impedance Z_3 is derived and presented in Fig.2.

$$\frac{1}{s \left(C_0 + \frac{C_m}{1 + C_m s (R_m + L_m s)} \right)}$$

Fig. 2. The impedance of the crystal unit.

Regardless of the initial system description, the transformed system can be easily generated by transforming the description in the specification list. Even more, any of the constative elements can be replaced with one or more connected elements and thus simplified analysis of very complex systems.

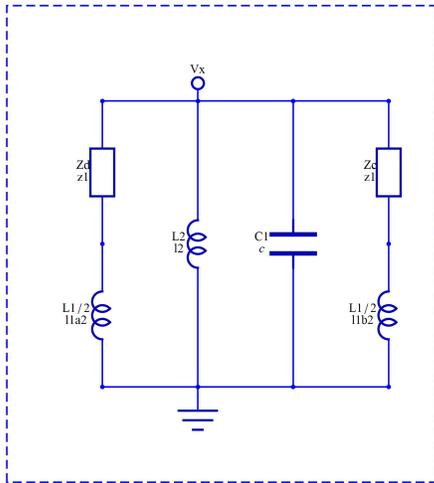


Fig. 3. Schematic of the impedance with two crystal units, three inductors and one capacitor.

Using the derived impedance of the crystal unit in

combination with several LC components, we can make another impedance, such as that presented in Fig. 3:

$$V / (V / (Z_D + s (L_1/2)) + V / (1 / (s C_1)) + V / (s L_2) + V / (Z_C + s (L_1/2)))$$

Suppose that the applied voltage is V . The resultant impedance is derived and presented in Fig.4.

$$\frac{1}{\frac{1}{L_2 s} + C_1 s + \frac{L_1 s}{2} + \frac{2}{s \left(C_0 + \frac{C_m}{1 + C_m s (R_m + L_m s)} \right)}}$$

Fig. 4. The impedance of two crystal units, three inductors and one capacitor.

For the example filter case, we can draw impedance presented in Fig. 5:

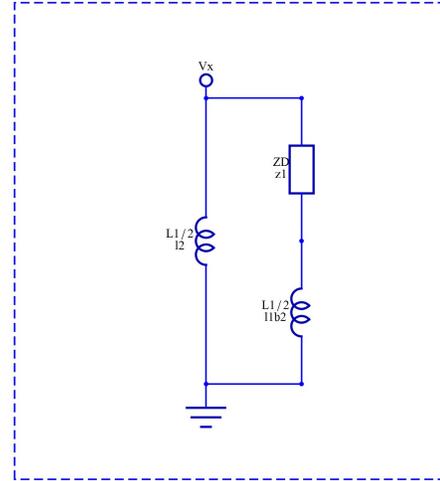


Fig. 5. Schematic of the impedance with one crystal unit and two inductors.

$$Z_5 = V / (V / (s (L_1/2)) + V / (Z_D + s (L_1/2)))$$

Suppose that the applied voltage is V . The resultant impedance Z_5 is derived and presented in Fig.6.

$$\frac{1}{\frac{2}{L_1 s} + \frac{L_1 s}{2} + \frac{1}{s \left(C_0 + \frac{C_m}{1 + C_m s (R_m + L_m s)} \right)}}$$

Fig. 6. The impedance of one crystal unit and two inductors.

The overall filter is generated using the new specification list, and illustrated in Fig. 7.

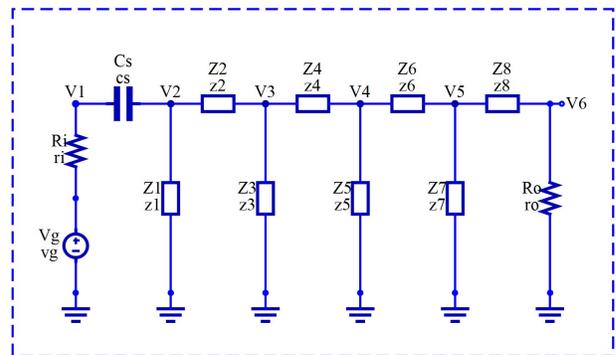


Fig. 7. Schematic of the overall filter with crystal units.

The impedances **Z3** and **Z5** are generated as separate lists.

Now is a time to transform schematic specification list into system of equations:

```
eqs = {
  Ir == (Vg - V1)/Ri,
  Is == (V1 - V2)/(1/(s Cs)),
  I1 == V2/Z1,
  I2 == (V2 - V3)/Z2,
  I3 == V3/Z3,
  I4 == (V3 - V4)/Z4,
  I5 == V4/Z5,
  I6 == (V4 - V5)/Z6,
  I7 == V5/Z7,
  I8 == (V5 - V6)/Z8,
  Io == V6/Ro,
  Ir == Is,
  Is == I1 + I2,
  I2 == I3 + I4,
  I4 == I5 + I6,
  I6 == I7 + I8,
  I8 == Io
};
```

Each element (impedance, resistor, inductor, and capacitor) is represented using Ohm's law as a ratio of the voltage across that element and the current through the element. In addition, each node is used to specify the Kirchhoff's Current Law.

Therefore, it is not required to set-up any specific type of analysis; just the simplest laws are used. Of course, the number of equations can be extremely large, but the computer system is solving the system of equations and there are no limits for regular systems that we can find in practice.

The newest version of SchematicSolver has many new elements, such as resistors, capacitors, inductors, current and voltage sources, or operating amplifiers and semiconductor devices.

III. SOLVING ELECTRICAL SYSTEMS

The solving of any system is a response of any kind with respect to inputs. The formal procedure is also very simple.

The first step is to define the variables that we have in our system:

```
vars = {V1, V2, V3, V4, V5, V6, Ir, Is, I1, I2, I3,
  I4, I5, I6, I7, I8, Io};
```

Any impedance can be replaced with single element or very complex impedance (such as derived **Z3** and **Z5**) and equivalent impedance of the crystal unit **ZA=Zcu**, **ZB=Zcu**):

```
subsZ = {Z1->s (L1/2), Cs->Ct, Z8->1/(s Ct),
  Z7->s (L1/2), Z3->Z3e, Z5->Z5e, Z2->ZA, Z4->ZB, Z6->ZA};
```

Next commands are used for solving the system of equations:

```
sol = Solve[eqs, vars];
tf1a = (V6/Vg /. sol) /. subsZ;
tf1b = tf1a // Together
Factor[Numerator[tf1b]]
Factor[Denominator[tf1b]]
```

The first line is solving the system of equations in terms of all variables. The number of equations should be equal to the number of variables.

The second command line is deriving the transfer function of the output node **V6** in with respect to the input voltage **Vg**,

using the solutions **sol** and substitution roles **subsZ**. The third command line is used to simplify the result in a form of ratio of two polynomials. The last two command lines are extraction of the numerator and denominator of the transfer function, and the results are presented in Figs. 8 and 9.

Factor[Numerator[tf1b]]

$$\{Ct^2 L1^3 L2 Ro s^9 (C\theta + Cm + C\theta Cm Rm s + C\theta Cm Lm s^2)^3 (2 + 2 Cm Rm s + C\theta L1 s^2 + Cm L1 s^2 + 2 Cm Lm s^2 + C\theta Cm L1 Rm s^3 + C\theta Cm L1 Lm s^4)^2\}$$

Fig. 8. Numerator of the transfer function.

Factor[Denominator[tf1b]]

$$\{32 + 32 Ct Ri s + 16\theta Cm Rm s + 32 Ct Ro s + 5595 \dots + C\theta^4 C1 Cm^5 Ct^2 L1^5 L2 Lm^5 Ro s^{23} + 5 C\theta^5 C1 Cm^5 Ct^2 L1^5 L2 Lm^4 Ri Rm Ro s^{23} + C\theta^5 C1 Cm^5 Ct^2 L1^5 L2 Lm^5 Ri Ro s^{24}\}$$

Fig. 9. Denominator of the transfer function.

The first derivative (see Fig. 10) with respect to the circular frequency can be used to drive the relation between filter elements. Numerator has more than 600,000 terms, and it is obvious that manual derivation is impossible.

```
fd1 = Numerator[Together[D[ntf1s/w], w]]];
```

fd1

$$\{-2 (\dots 629 545 \dots + 1\theta c^2 C\theta^{20} Cm^{20} Ct^8 L^{24} Lm^{18} R^6 Rm^2 w^{91} + c^2 C\theta^{20} Cm^{20} Ct^8 L^{24} Lm^{20} R^6 w^{93})\}$$

Fig. 10. The first derivative of the transfer function.

Using normalization of the frequency, the response is simplified, but it is still with more than 600,000 terms.

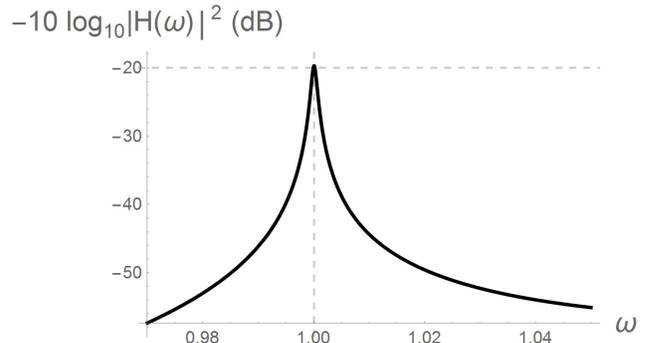


Fig. 11. Magnitude response versus the normalized frequency $\omega_n=10^8$ rad/s.

The magnitude response is plotted with respect to the normalized frequency (see Fig. 11).

It is not visible, but the maximum is not exactly at $\omega=10^8$ rad/s. To find the value of the capacitance **C1**, it is necessary to find closed-form expression of the transfer function with symbolic value of the capacitance **C1**, substitute all filter values, and generate equation from the condition to have zero valued the first derivative with respect to the frequency

```
fa = (ntf1s/df1s) /. subsV /. subsCu // Together;
```

```
Solve[Numerator[D[fa, w] // Together] == 0, c];
% /. w -> 1 // N
```

The exact value of the capacitance C_1 is a solution for the frequency equal to the normalized frequency. Two results are obtained:

```
{c->0.000634251}, {c->0.000348746600643066}}
```

The second solution is the exact value. Without normalization, the exact value ($C_{1,\text{exact}}$) of the capacitance is slightly larger than that obtained using numeric approximate approach ($C_{1,\text{numer}}$)

$$C_{1,\text{exact}} = 3.48746600643066 \cdot 10^{-12} \text{ F}$$

$$C_{1,\text{numer}} = 3.486 \cdot 10^{-12} \text{ F}$$

Many other analyses can be programmed using combined numeric and symbolic processing. In some cases it is useful to view intermediate numeric results and to avoid symbolic computations prior to the decision to wait much more for the exact results.

In the next example, we are using specified values of filter elements, and we are developing the relation between two element values in the case that one element can be changed with respect to the exact value. Suppose that inductances L_1 and L_2 ($L_1=L_2=L$) can be within 10% of the nominal value $L=9\mu\text{H}$. The maximum of the magnitude should be at $\omega=10^8$ rad/s. The first step is to find the expression of the frequency in terms of all element values (numerator of the first derivative of the magnitude response)

```
fd1=Numerator[Together[D[ntf1s/df1s, w]]];
```

The result has more than 400,000 terms.

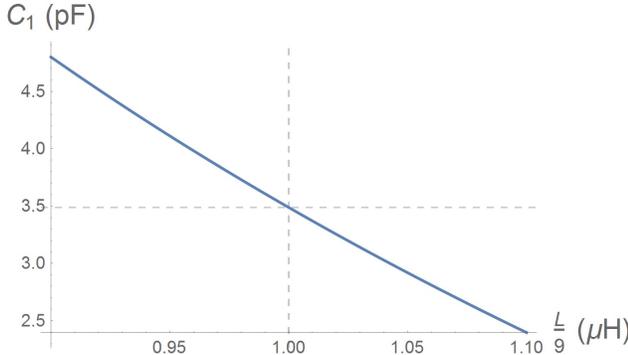


Fig. 12. The value of the capacitance C_1 for 10% tolerances of L so that the magnitude response has the maximum at $\omega=10^8$ rad/s.

From the condition that the first derivative is equal to 0 at the specified frequency ($\omega=10^8$ rad/s), with the substitution of some known element values ($R_i=R_o=1233 \Omega$, $C_i=3.486$ pF, $R_m=199.7 \Omega$, $L_m=21.19 \mu\text{H}$, $C_m=83.02$ fF, $C_0=1.743$ pF) we can derive the relation between C_1 and L . Again, we cannot present the derived relation due to the large number of terms,

but we can plot the values of C_1 in terms of L when the maximum of the magnitude is exact (see Fig. 12).

The derived expression (presented in Fig. 13.) can be used for tuning the capacitance for measured values of L .

$$-6.97234 + \frac{11.1111}{\text{Ln}} - 0.546902 \text{Ln} -$$

$$0.082451 \text{Ln}^2 - 0.0165714 \text{Ln}^3 -$$

$$0.00395019 \text{Ln}^4 - 0.00103224 \text{Ln}^5$$

Fig. 13. The approximate value of the capacitance C_1 in terms of the normalized inductance $L_n=L/9 \mu\text{H}$ so that the magnitude response has the maximum at $\omega=10^8$ rad/s.

The presented methodology illustrates how we can combine the symbolic derivations that are inappropriate for manual derivation in order to discover relations required for manufacturing and tuning.

It is important to notice that we have started with drawing the schematic using the SchematicSolver's GUI in computer algebra system, and the whole procedure can be automated in a form of notebook using Wolfram Language.

IV. CONCLUSION

The software described in this paper presents referenced document that contains all required knowledge and procedures for modeling, analyzing, and designing of quartz crystals filters. After drawing the schematic using appropriate GUI, the rest of the procedure can be automated to find the responses or values of the filter elements.

ACKNOWLEDGMENT

This work was supported by the Ministry of Education and Science of Serbia under Grant TR 32048.

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