WAVELET AND SCALING FUNCTIONS OF TWO-BAND ORTHONORMAL RATIONAL FILTER BANKS

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Abstract – *In this paper, we consider extension of the ordinary dyadic wavelet transform, implemented by iteration of the two-band orthonormal rational filter banks on the lowpass branch, that provide a flexible orthogonal tiling of the time-frequency plane. We design the rational two-band orthonormal IIR filter banks starting from the fifth order halfband Butterworth filter bank and generate pertaining wavelet and scaling functions. We show the appropriate time-frequency plane tiling. The same graphs are depicted for the dyadic wavelet transform implemented by the iterations of the fifth order Butterworth filter bank.*

1. INTRODUCTION

In this paper, we consider an extension of the ordinary dyadic wavelet transform implemented by iteration of the two-band orthonormal rational filter banks on the lowpass branch [1] - [3] that provide a flexible orthogonal tiling of the time-frequency plane. The rational orthonormal IIR filter bank with cutoff frequency $fc = 0.5/n$, $n \in$ is constructed from the QMF two-band IIR filter bank using algorithm proposed in [4] and [5].

The choice of the proper tiling of the time-frequency plane may dramatically improve performance in several applications, such as audio coding, analysis, synthesis and denoising. By adapting the frequency bands to the signal, we can enhance energy compaction and improve the coding efficiency. Choosing arbitrary cutoff frequency an entire class of nonuniform filter banks can be derived.

Two-band rational FIR filter banks have been examined e.g. in [6] – [8], while the IIR orthonormal wavelet filter banks have been less studied in general [9], [10].

In this paper, we design the rational two-band orthonormal IIR filter banks, using algorithm for construction of IIR complementary filter pairs with an adjustable crossover frequency given in [4] and [5], starting from the fifth order halfband Butterworth filter bank and generate pertaining wavelet and scaling functions. It the end, we show the appropriate time-frequency plane tilings. For the comparison, the same graphs are depicted for the dyadic wavelet transform implemented by the iterations of the initial fifth order Butterworth filter bank.

2. TWO-BAND RATIONAL IIR FILTER BANKS

An odd-order IIR filter pair $\boldsymbol{H}(z)$ can be implemented as a parallel connection of two all-pass branches $A_0(z)$ and $A_1(z)$:

$$\mathbf{H}(z) = \begin{bmatrix} H_{LP}(z) \\ H_{HP}(z) \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} A_0(z) + A_1(z) \\ A_0(z) - A_1(z) \end{bmatrix}. \quad (1)$$

where $H_{LP}(z)$ denotes low-pass and $H_{HP}(z)$ high-pass filter. This filter pair called also two-channel filter bank, is power complementary and all-pass complementary [11].

For dividing the basis band in two equal sub-bands, the crossover frequency of the filter pair should be placed at the middle of the band, and both filters, $H_{LP}(z)$ and $H_{HP}(z)$, should be half-band filters satisfying the well-known symmetry conditions [11]. The crossover frequency of the filter pair is also the 3dB cut-off frequency for $H_{LP}(z)$ and $H_{HP}(z)$, i.e., $f_c^{HB} = 0.25$.

The all-pass branches $A_0(z)$ and $A_1(z)$, are implemented as a cascade connection of the first-order and the second-order sections. In the case of a minimum phase half-band filter, all poles are placed on the imaginary axes and one of them is placed at the origin. Therefore, the transfer function of the half-band filter pair $\mathbf{H}^{HB}(z)$ can be expressed as follows:

$$\mathbf{H}^{\mathbf{HB}}(z) = \frac{1}{\sqrt{2}} \left[\prod_{i=2,4,\ldots}^{(N+1)/2} \frac{\beta_i^{HB} + z^{-2}}{1 + \beta_i^{HB} z^{-2}} \pm z^{-1} \prod_{i=3,5,\ldots}^{(N+1)/2} \frac{\beta_i^{HB} + z^{-2}}{1 + \beta_i^{HB} z^{-2}} \right] \quad (2)$$

where $\beta_i^{HB} < \beta_{i+1}^{HB}$. Here the constant $\beta_i^{HB} = \left| r_i^{HB} \right|^2$ denotes the square module of the half-band filter pole r_i^{HB}.

When the crossover frequency of the filter pair is moved from the point $f_c^{HB} = 0.25$ to some other position in the range $0 < f_c < 0.5$, the transfer function of the transformed filter pair $\mathbf{H}^{HB}(z)$ can be expressed [4], [5] in the form:

$$\mathbf{H}(z) = \frac{1}{\sqrt{2}} \left[\prod_{i=2,4,\ldots}^{(N+1)/2} \frac{\beta_i + \alpha(1+\beta_i) z^{-1} + z^{-2}}{1 + \alpha(1+\beta_i) z^{-1} + \beta_i z^{-2}} \right.$$
$$\left. \pm \frac{\alpha_1 + z^{-1}}{1 + \alpha_1 z^{-1}} \prod_{i=3,5,\ldots}^{(N+1)/2} \frac{\beta_i + \alpha(1+\beta_i) z^{-1} + z^{-2}}{1 + \alpha(1+\beta_i) z^{-1} + \beta_i z^{-2}} \right], \ \beta_i < \beta_{i+1} . (3)$$

Table 1 presents the formulae for computing the constants α and β_i of equation (3) using the constants β_i^{HB} of the start-up half-band filter from (2) and desired crossover frequency f_c as given in [4] and [5]. These formulae are applicable also for odd-order Butterworth filters when implemented as a parallel connection of two all-pass branches $A_0(z)$ and $A_1(z)$, [12].

The 3-dB cutoff frequency of a single filter is equivalent to the 3-dB crossover frequency of a complementary filter pair.

TABLE I - COMPUTATION OF CONSTANTS

First-order Section	$\alpha_1 = -\left(1-\tan\left(\pi f_{3dB}\right)\right)/\left(1+\tan\left(\pi f_{3dB}\right)\right)$	
Second-order Section	$\alpha = -\dfrac{1-\left(\tan\left(\pi f_{3dB}\right)\right)^2}{1+\left(\tan\left(\pi f_{3dB}\right)\right)^2}$	$\beta_i = \dfrac{\beta_i^{HB}+\alpha_1^2}{\beta_i^{HB}\alpha_1^2+1}$

According to this algorithm, we design two IIR filter banks whose crossover frequency are moved from $f_c^{HB} = 0.25$ to of the halfband pair to $f_c = 0.5/3$ and $f_c = 0.5/4$ while still retaining the attenuation properties of the initial fifth order halfband Butterworth filters. Transfer functions and magnitude characteristics of the initial and the resulting filter pairs are depicted in figures 1 and 2 respectively.

In orthonormal wavelet transform implementation using lowpass iteration of the halfband filter banks [13] – [15], the most desirable filter property is high regularity order which represents the numbers of zeros at half the sampling frequency of the lowpass filter of the analysis bank [16]. Also, the regularity order denotes the implemented wavelet transform capability of detecting signal discontinuities of the higher order [15], [17]. In figure 3, the zero-pole plots of the initial and the designed lowpass filters are shown. We notice that the applied procedure for changing the crossover frequency preservers the regularity order of the initial filter. The IIR filters with crossover frequencies $f_c = \frac{1}{6}$ and $f_c = \frac{1}{8}$ have five zeros at $z = -1$ as the initial halfband Butterworth filter.

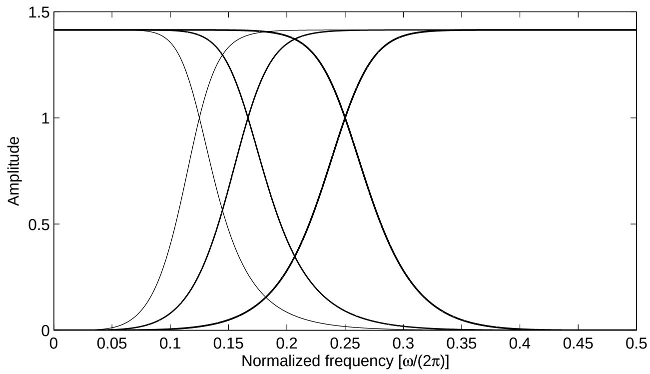

Fig. 1. *Transfer functions*

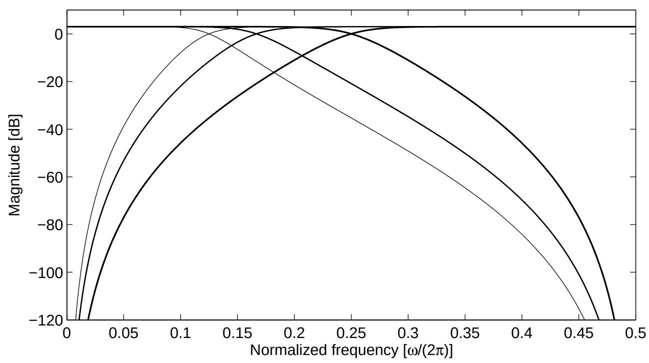

Fig. 2. *Magnitude characteristics*

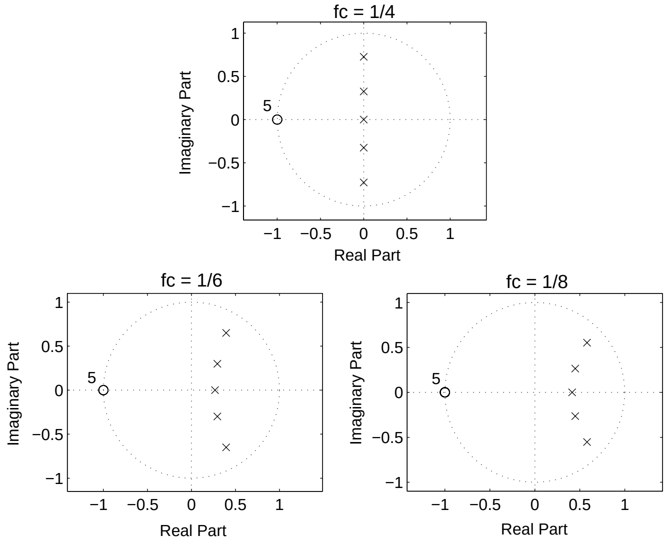

Fig. 3. *Zero-pole positions of the lowpass filters*

3. WAVELET TRANSFORM IMPLEMENTATION

The lowpass branch iteration of the filter banks generates equivalent bandpass filters [16] of the form

$$\Psi^i(z) = H_0(z)H_0(z^2)\cdots H_0(z^{2^{i-2}})H_1(z^{2^{i-1}})\,. \qquad (4)$$

Letting $i \to \infty$ gives the "mother wavelet" $\psi(t)$:

$$\psi(t) = \lim_{i\to\infty} \psi^i_{\lfloor 2^i t \rfloor}\,, \qquad (5)$$

where ψ_n^i is the impulse response of $\Psi^i(z)$. Impulse response ϕ_n^i of the equivalent lowpass filter of the form

$$\Phi^i(z) = H_0(z)H_0(z^2)\cdots H_0(z^{2^{i-2}})H_0(z^{2^{i-1}}) \qquad (6)$$

is reffered as scaling function after i iterations.

The fourth order iteration of the two-band filter banks realising orthonormal wavelet decomposition and synthesys are displayed in figures 4 and 5 respectively. If the input signal is the unit impulse, the highpass outputs of the analysis filter bank iteration and the corresponding inputs of the synthesis filter bank iteration, denoted with $\psi_{i,0}$, represent the wavelet functions after i iteration. In the same time, $\phi_{4,0}$ represents the scaling function after four iterations. Orthonormality property is attained by employing the anticausal versions of the wavelet decomposition filters in the synthesis part [18], [19], [13] - [15].

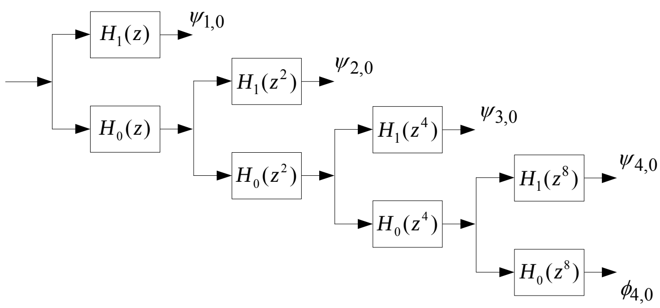

Fig. 4. *Filter bank implementation of the wavelet transform*

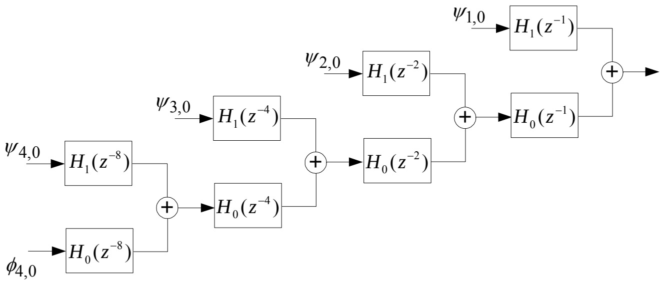

Fig. 5. *Filter bank implementation of the inverse wavelet transform*

Implementation of the wavelet transform using rational two-band filter bank provides tiling of the time-frequency plane in accordance with the chosen cutoff filter pair frequency. Equivalent bandpass filters' transfer functions i.e. wavelet spectra $\left|\psi_{i,0}\right|$, $i = 1,2,3,4$ and the lowpass filters' i.e.scalinig functions' spectra $\left|\phi_{4,0}\right|$, of the iterated analysis filter banks, as well as their time representations, are ilustrated in figures from 6 to 11.

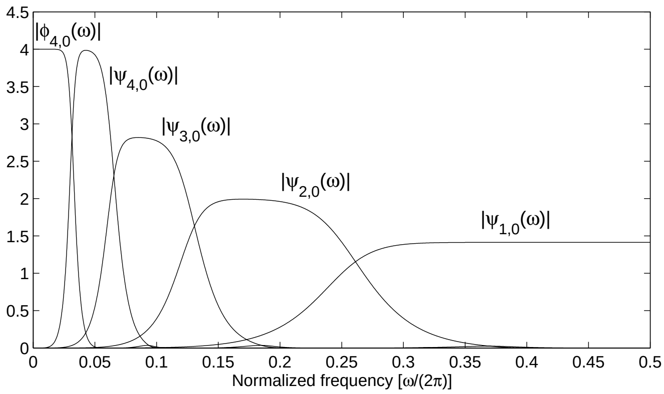

Fig. 6. *Wavelet and scaling functions spectra,* $f_c = \frac{1}{4}$

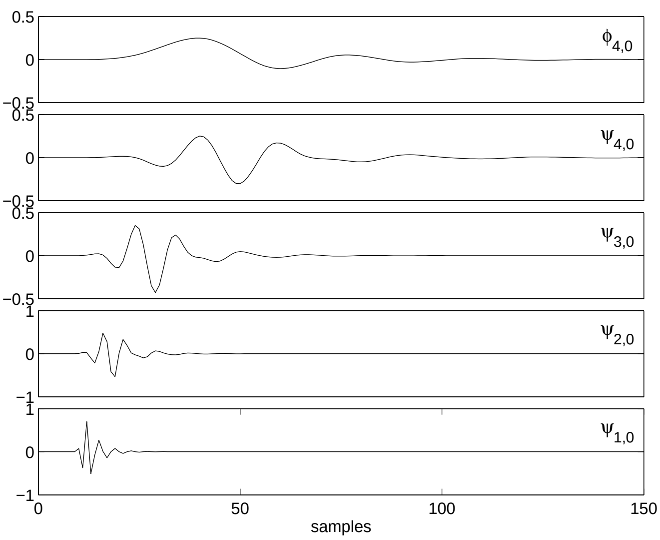

Fig. 7. *Wavelet and scaling functions,* $f_c = \frac{1}{4}$

Figure 12 depicts the time frequency tiling for the examined dyadic and rational filter banks with cutoff frequency of $f_c = \frac{1}{6}$ and $f_c = \frac{1}{8}$.

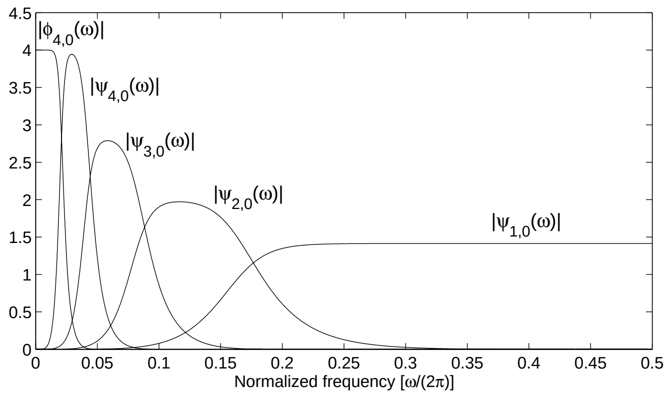

Fig. 8. *Wavelet and scaling functions spectra,* $f_c = \frac{1}{6}$

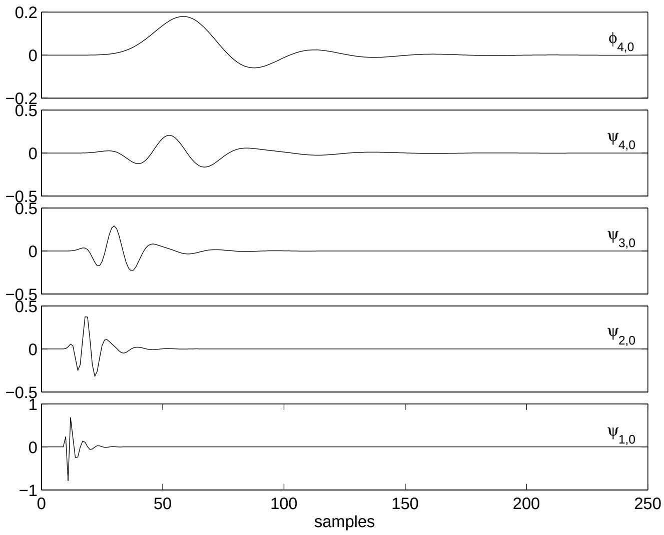

Fig. 9. *Wavelet and scaling functions,* $f_c = \frac{1}{6}$

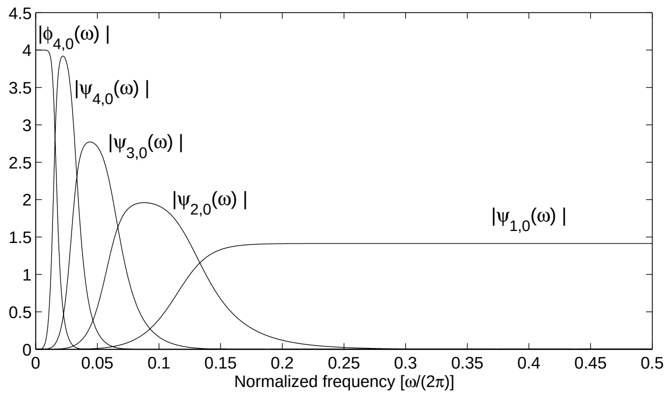

Fig. 10. *Wavelet and scaling functions spectra,* $f_c = \frac{1}{8}$

4. CONCLUSION

In this paper, we have demonstrated the discrete-time orthonormal wavelets with nonoctave spaced frequency bands generated by iteration of the two-band orthonormal rational IIR filter banks derived from the fifth order halfband Butterworth filter banks. The most important property of the derived IIR rational filter banks is that they preserve the filter and regularity orders of the initial halfband IIR filter bank.

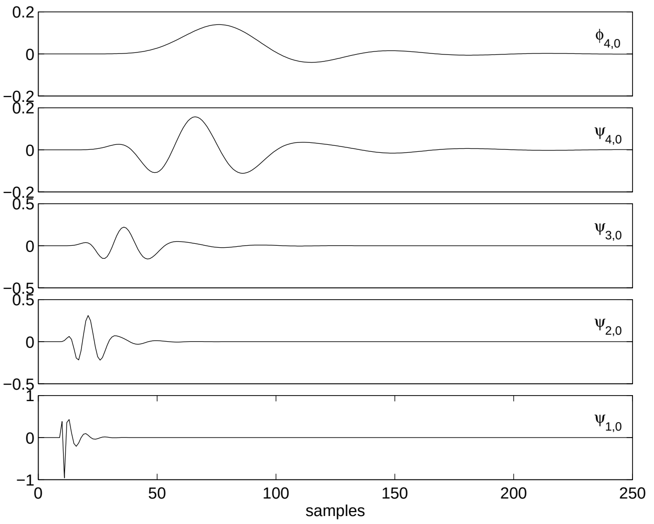

Fig. 11. *Wavelet and scaling functions,* $f_c = \frac{1}{8}$

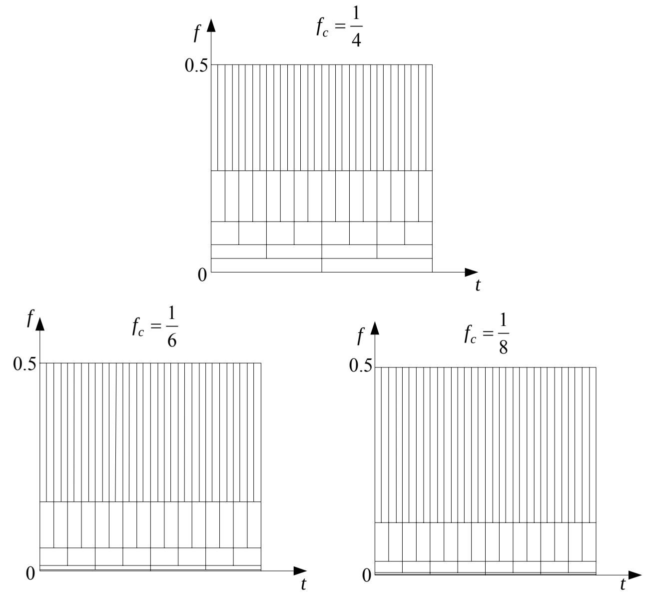

Fig.12 *The time-frequency plane tiling*

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