The application of Pythagorean trials to trigonometry and hyperbolic functions and their use in technical sciences

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Abstract **— To calculate the elements of triangles, it is necessary to establish a connection with the angles that define them, but usually we encounter the specific angles values 30°, 45° and 60° (sometimes also 75° or 15°). This limitation led to the examination of the application of Pythagorean triples in trigonometry, after which the question arose whether and in what way they can be applied to hyperbolic functions. Several classical formulas have been chosen that have their own interesting history, but in this paper some relations are observed, which aim to contribute to a shortening of the procedures used for calculating the functions of argument. Thus, they would find their application in other sciences, but also in specific areas of everyday human activities.**

Index Terms **— Pythagorean triplets, trigonometry, hyperbolic functions, application.**

I. Introduction

The word *trigonometry* is derived from the ancient Greek words *trigonon* (triangle) and *metron* (measure). The name itself testifies to the fact that she initially dealt with the problem of measuring a triangle (its sides and angles). Its roots are deep in the past, in the ancient times, in Babylon, Egypt, India, Greece [7]. The modern notion of trigonometry, apart from calculating the elements of a triangle, deals with establishing the functions of the real (even complex) argument that define them and represent the basis for showing pheno-mena and processes in modern technical practice.

In mathematics, hyperbolic functions are defined in a way similar to trigonometric functions [3, 8, 10]. Just as a trigonometric function can be defined for, or on, a circle, a hyperbolic function is defined for a hyperbola. In trigonometry, sine (sin*x*), cosine (cos*x*) and other functions appear. Similarly, hyperbolic functions are used: hyperbolic sine (sh*x*, or sinh*x*), hyper-bolic cosine (ch*x*, or cosh*x*), hyperbolic tangent (th*x*, or tanh*x*), hyperbolic cotangent (cth*x*, or coth*x*), hyperbolic secant (sch*x*, or sech*x*), hyperbolic cosecant (csch*x*, or cosesh*x*). Their inverse functions are prefixed with ar(ea), unlike trigonometric arc(us). Just as in trigonometry the coordinates of points on the unit circle are (cos*x*, sin*x*), in hyperbolic functions pair (ch*x*, sh*x*) forms the right half of an equilateral hyperbola. These functions can be solutions of various linear differential equations, Laplace equation, etc., and are defined through algebraic expressions, that include the exponential function (e^x) and its negative exponential function (e^{-x}), where e is Euler number.

Hyperbolic sine is the odd part of exponential function, and its algebraic expression is:

$$\mathrm{sh}x = \frac{e^x - e^{-x}}{2}. \tag{1}$$

Hyperbolic cosine represents the even part of exponential functions, and its algebraic expression is:

$$\mathrm{ch}x = \frac{e^x + e^{-x}}{2}. \tag{2}$$

The hyperbolic tangent is defined as follows:

$$\mathrm{th}x = \frac{\mathrm{sh}x}{\mathrm{ch}x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \tag{3}$$

and the cotangent to the following:

$$\mathrm{cth}x = \frac{\mathrm{ch}x}{\mathrm{sh}x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}. \tag{4}$$

The hyperbolic secant is represented as:

$$\mathrm{sch}x = \frac{1}{\mathrm{ch}x} = \frac{2}{e^x + e^{-x}}, \tag{5}$$

and the cosecant as:

$$\mathrm{csch}x = \frac{1}{\mathrm{sh}x} = \frac{2}{e^x - e^{-x}}. \tag{6}$$

II. Materials and methods - Pythagorean triples

An ordered triple of natural numbers (a, b, c) is called a *Pythagorean triple*, if it is a solution to the equation:

$$a^2 + b^2 = c^2. \tag{7}$$

The theorem itself expresses the relationship that exists between the three sides of a right triangle in Euclidean geometry. If a and b are legs, and c is the hypotenuse of a right triangle, the equality stated in (7) applies. They were named after the ancient Greek mathematician and philosopher Pythagoras, who is traditionally associated with their discovery, although today it is certain that it was known long before him, and that he was actually just an intermediary between the knowledge that came from the East to the Greeks. In order to come up with formulas that would enable faster and simpler obtaining of the values needed for the calculation of trigonometric functions, we used three formulas (8-10) [1] to obtain the values of Pythagorean triples. The symbols p and q indicate the variables that make up the data on the basis of which the measurements of the legs (catheti) and hypotenuses are obtained, and the angle of inclination is the angle between the hypotenuse and one of the legs of the right triangle.

$$p\ ;\ a = 2p + 1\ ;\ b = 2p^2 + 2p\ ;\ c = 2p^2 + 2p + 1, \tag{8}$$

$$p\ ;\ a = 2p\ ;\ b = p^2 - 1\ ;\ c = p^2 + 1, \tag{9}$$

$$p\ ;\ q\ ;\ a = 2pq\ ;\ b = p^2 - q^2\ ;\ c = p^2 + q^2, \tag{10}$$

 ISBN 978-86-7466-969-3

where:

p – the first variable, q – the second variable,
$p > q$ (condition),
a, b – legs of a right triangle, c – hypotenuse.

III. Main Results

The angles are marked with the Pythagorean triples as follows:

$$\alpha = (a_1, b_1, c_1),\ \beta = (a_2, b_2, c_2),\ \gamma = (a_3, b_3, c_3).$$

Then, for the sum of angles we have:

$$\boldsymbol{\gamma = \alpha + \beta} = (a_1 \cdot a_2 - b_1 \cdot b_2,\ a_1 \cdot b_2 + a_2 \cdot b_1,\ c_1 \cdot c_2). \tag{11}$$

The proof (based on the geometry presented in Figure 1):

1. OA = a_1 AB = b_1 OB = c_1
2. OC = a_2 CD = b_2 OD = c_2

The similarity of triangles: ΔOAB ~ ΔOFC ~ ΔDGC

OA = a_1 OF DG
AB = b_1 FC GC
OB = c_1 OC = a_2 DC = b_2

$$\Rightarrow OF = \frac{OA \cdot OC}{OB} = \frac{a_1 \cdot a_2}{c_1}$$

$$FC = \frac{AB \cdot OC}{OB} = \frac{b_1 \cdot a_2}{c_1},\ DG = \frac{OA \cdot DC}{OB} = \frac{a_1 \cdot b_2}{c_1}.$$

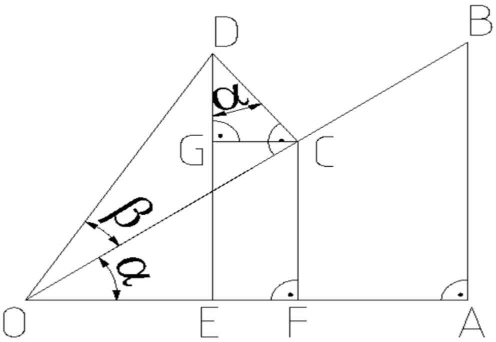

Figure 1. Right triangle which has one angle that is equal to sum of angles **α + β**

$$GC = \frac{AB \cdot DC}{OB} = \frac{b_1 \cdot b_2}{c_1}$$

$$\Rightarrow OE = OF - FE = OF - GD =$$

$$= \frac{a_1 \cdot a_2}{c_1} - \frac{b_1 \cdot b_2}{c_1} = \frac{a_1 \cdot a_2 - b_1 \cdot b_2}{c_1}$$

$$ED = EG + GD = FC + GD =$$

$$\frac{b_1 \cdot a_2}{c_1} + \frac{a_1 \cdot b_2}{c_1} = \frac{a_1 \cdot b_2 + b_1 \cdot a_2}{c_1}$$

$OD = c_2$.

When the last three equations are multiplied by the value c_1, the required formula is obtained:

$$\gamma = \alpha + \beta = (a_1 \cdot a_2 - b_1 \cdot b_2,\ a_1 \cdot b_2 + a_2 \cdot b_1,\ c_1 \cdot c_2),$$

which makes the proof complete.

For the subtraction of angles (the proof is done analogously):

$$\boldsymbol{\gamma = \alpha - \beta} = (a_1 \cdot a_2 + b_1 \cdot b_2,\ b_1 \cdot a_2 - b_2 \cdot a_1,\ c_1 \cdot c_2) \tag{12}$$

For a double angle, by substituting the value in formula (11), we get:

$$\boldsymbol{\gamma = 2\alpha} = (a^2 - b^2,\ 2ab,\ c^2). \tag{13}$$

For the half angle (the proof is done analogously):

$$\gamma = \frac{\boldsymbol{\alpha}}{\boldsymbol{2}} = \left(\sqrt{\frac{c+a}{2}}\ ,\ \sqrt{\frac{c-a}{2}}\ , \sqrt{c} \right). \tag{14}$$

When the angles are marked with the Pythagorean triples:

$$\alpha = (1, \frac{b}{c}, \frac{a}{c}) = (1, \frac{b_1}{c_1}, \frac{a_1}{c_1});$$

$$\beta = (1, \frac{b_2}{c_2}, \frac{a_2}{c_2}),$$

then for the sum of angles we have:

$$(\alpha + \beta) = \left(1, \frac{b_1 \cdot a_2 + a_1 \cdot b_2}{c_1 \cdot c_2}, \frac{a_1 \cdot a_2 + b_1 \cdot b_2}{c_1 \cdot c_2}\right). \tag{15}$$

Proof:

$$\textbf{1)}\quad \mathbf{sh}(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \mathbf{sh}\,\boldsymbol{\alpha}\,\mathbf{ch}\,\boldsymbol{\beta} + \mathbf{ch}\,\boldsymbol{\alpha}\,\mathbf{sh}\,\boldsymbol{\beta}$$

$$\frac{e^{\alpha+\beta} - e^{-(\alpha+\beta)}}{2} = \frac{e^{\alpha} - e^{-\alpha}}{2} \cdot \frac{e^{\beta} + e^{-\beta}}{2} + \frac{e^{\alpha} + e^{-\alpha}}{2} \cdot \frac{e^{\beta} - e^{-\beta}}{2}$$

$$\frac{e^{\alpha} \cdot e^{\beta} - e^{-\alpha} \cdot e^{-\beta}}{2} = \frac{b_1}{c_1} \cdot \frac{a_2}{c_2} + \frac{a_1}{c_1} \cdot \frac{b_2}{c_2}$$

$$\frac{\frac{a_1+b_1}{c_1} \cdot \frac{a_2+b_2}{c_2} - \frac{a_1-b_1}{c_1} \cdot \frac{a_2-b_2}{c_2}}{2} = \frac{a_1 \cdot b_2 + a_2 b_1}{c_1 c_2}$$

$$\frac{a_1 \cdot a_2 + a_1 b_2 + a_2 b_1 + b_1 b_2 - a_1 a_2 + a_1 b_2 + a_2 b_1 - b_1 b_2}{2 c_1 c_2} = \frac{a_1 \cdot b_2 + a_2 b_1}{c_1 c_2}$$

$$\frac{2 a_1 \cdot b_2 + 2 a_2 b_1}{2 c_1 c_2} = \frac{a_1 \cdot b_2 + a_2 b_1}{c_1 c_2}.$$

$$\textbf{2)}\quad \mathbf{ch}(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \mathbf{ch}\,\boldsymbol{\alpha}\,\mathbf{ch}\,\boldsymbol{\beta} + \mathbf{sh}\,\boldsymbol{\alpha}\,\mathbf{sh}\,\boldsymbol{\beta}$$

$$\frac{e^{\alpha+\beta} + e^{-(\alpha+\beta)}}{2} = \frac{e^{\alpha} + e^{-\alpha}}{2} \cdot \frac{e^{\beta} + e^{-\beta}}{2} + \frac{e^{\alpha} - e^{-\alpha}}{2} \cdot \frac{e^{\beta} - e^{-\beta}}{2}$$

$$\frac{e^{\alpha} \cdot e^{\beta} + e^{-\alpha} \cdot e^{-\beta}}{2} = \frac{a_1}{c_1} \cdot \frac{a_2}{c_2} + \frac{b_1}{c_1} \cdot \frac{b_2}{c_2}$$

$$\frac{\frac{a_1+b_1}{c_1} \cdot \frac{a_2+b_2}{c_2} + \frac{a_1-b_1}{c_1} \cdot \frac{a_2-b_2}{c_2}}{2} = \frac{a_1 \cdot a_2 + b_1 b_2}{c_1 c_2}$$

$$\frac{a_1 \cdot a_2 + a_1 b_2 + a_2 b_1 + b_1 b_2 + a_1 a_2 - a_1 b_2 - a_2 b_1 + b_1 b_2}{2 c_1 c_2} = \frac{a_1 \cdot a_2 + b_1 b_2}{c_1 c_2}$$

$$\frac{2 a_1 \cdot a_2 + 2 b_1 b_2}{2 c_1 c_2} = \frac{a_1 \cdot a_2 + b_1 b_2}{c_1 c_2}$$

$$\frac{2 \cdot (a_1 \cdot a_2 + b_1 b_2)}{2 c_1 c_2} = \frac{a_1 \cdot a_2 + b_1 b_2}{c_1 c_2}$$

$$\frac{a_1 \cdot a_2 + b_1 b_2}{c_1 c_2} = \frac{a_1 \cdot a_2 + b_1 b_2}{c_1 c_2}$$

$$\textbf{3)}\quad \mathbf{th}(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \frac{\mathbf{th}\,\boldsymbol{\alpha} + \mathbf{th}\,\boldsymbol{\beta}}{\mathbf{1} + \mathbf{th}\,\boldsymbol{\alpha} \cdot \mathbf{th}\,\boldsymbol{\beta}}$$

$$\frac{e^{\alpha+\beta} - e^{-(\alpha+\beta)}}{e^{\alpha+\beta} + e^{-(\alpha+\beta)}} = \frac{\frac{e^{\alpha} - e^{-\alpha}}{e^{\alpha} + e^{-\alpha}} + \frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta}}}{1 + \frac{e^{\alpha} - e^{-\alpha}}{e^{\alpha} + e^{-\alpha}} \cdot \frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta}}}$$

$$\frac{e^{\alpha} \cdot e^{\beta} - e^{-\alpha} \cdot e^{-\beta}}{e^{\alpha} \cdot e^{\beta} + e^{-\alpha} \cdot e^{-\beta}} = \frac{\frac{b_1}{a_1} + \frac{b_2}{a_2}}{1 + \frac{b_1}{a_1} \cdot \frac{b_2}{a_2}}$$

$$\frac{\frac{a_1+b_1}{c_1} \cdot \frac{a_2+b_2}{c_2} - \frac{a_1-b_1}{c_1} \cdot \frac{a_2-b_2}{c_2}}{\frac{a_1+b_1}{c_1} \cdot \frac{a_2+b_2}{c_2} + \frac{a_1-b_1}{c_1} \cdot \frac{a_2-b_2}{c_2}} = \frac{\frac{a_2 b_1 + a_1 b_2}{a_1 a_2}}{\frac{a_1 a_2 + b_1 b_2}{a_1 a_2}}$$

 ISBN 978-86-7466-969-3

$$\frac{\frac{a_1a_2+a_1b_2+a_2b_1+b_1b_2-a_1a_2+a_1b_2+a_2b_1-b_1b_2}{c_1c_2}}{\frac{a_1a_2+a_1b_2+a_2b_1+b_1b_2+a_1a_2-a_1b_2-a_2b_1+b_1b_2}{c_1c_2}}=\frac{a_1b_2+a_2b_1}{a_1a_2+b_1b_2}$$

$$\frac{2a_1b_2+2a_2b_1}{2a_1a_2+2b_1b_2}=\frac{a_1b_2+a_2b_1}{a_1a_2+b_1b_2}$$

$$\frac{2(a_1b_2+a_2b_1)}{2(a_1a_2+b_1b_2)}=\frac{a_1b_2+a_2b_1}{a_1a_2+b_1b_2}$$

$$\frac{a_1b_2+a_2b_1}{a_1a_2+b_1b_2}=\frac{a_1b_2+a_2b_1}{a_1a_2+b_1b_2}$$

$$\textbf{4)}\quad \mathbf{cth}(\boldsymbol{\alpha}+\boldsymbol{\beta})=\frac{\mathbf{1+cth}\,\boldsymbol{\alpha}\cdot\mathbf{cth}\,\boldsymbol{\beta}}{\mathbf{cth}\,\boldsymbol{\alpha}+\mathbf{cth}\,\boldsymbol{\beta}}$$

$$\frac{e^{\alpha+\beta}+e^{-(\alpha+\beta)}}{e^{\alpha+\beta}-e^{-(\alpha+\beta)}}=\frac{1+\frac{e^{\alpha}+e^{-\alpha}}{e^{\alpha}-e^{-\alpha}}\cdot\frac{e^{\beta}+e^{-\beta}}{e^{\beta}-e^{-\beta}}}{\frac{e^{\alpha}-e^{-\alpha}}{e^{\alpha}-e^{-\alpha}}+\frac{e^{\beta}+e^{-\beta}}{e^{\beta}-e^{-\beta}}}$$

$$\frac{e^{\alpha}\cdot e^{\beta}+e^{-\alpha}\cdot e^{-\beta}}{e^{\alpha}\cdot e^{\beta}-e^{-\alpha}\cdot e^{-\beta}}=\frac{1+\frac{a_1}{b_1}\cdot\frac{a_2}{b_2}}{\frac{a_1}{b_1}+\frac{a_2}{b_2}}$$

$$\frac{\frac{a_1+b_1}{c_1}\cdot\frac{a_2+b_2}{c_2}+\frac{a_1-b_1}{c_1}\cdot\frac{a_2-b_2}{c_2}}{\frac{a_1+b_1}{c_1}\cdot\frac{a_2+b_2}{c_2}-\frac{a_1-b_1}{c_1}\cdot\frac{a_2-b_2}{c_2}}=\frac{\frac{b_1b_2+a_1a_2}{b_1b_2}}{\frac{a_1b_2+a_2b_1}{b_1b_2}}$$

$$\frac{\frac{a_1a_2+a_1b_2+a_2b_1+b_1b_2+a_1a_2-a_1b_2-a_2b_1+b_1b_2}{c_1c_2}}{\frac{a_1a_2+a_1b_2+a_2b_1+b_1b_2-a_1a_2+a_1b_2+a_2b_1-b_1b_2}{c_1c_2}}=\frac{a_1a_2+b_1b_2}{a_1b_2+a_2b_1}$$

$$\frac{2a_1a_2+2b_1b_2}{2a_1b_2+2a_2b_1}=\frac{a_1a_2+b_1b_2}{a_1b_2+a_2b_1}$$

$$\frac{2\cdot(a_1a_2+b_1b_2)}{2\cdot(a_1b_2+a_2b_1)}=\frac{a_1a_2+b_1b_2}{a_1b_2+a_2b_1}$$

$$\frac{a_1a_2+b_1b_2}{a_1b_2+a_2b_1}=\frac{a_1a_2+b_1b_2}{a_1b_2+a_2b_1}$$

For the subtraction of angles:

$$(\boldsymbol{\alpha}-\boldsymbol{\beta})=\left(\mathbf{1},\frac{\boldsymbol{b_1\cdot a_2-a_1\cdot b_2}}{\boldsymbol{c_1\cdot c_2}},\frac{\boldsymbol{a_1\cdot a_2-b_1\cdot b_2}}{\boldsymbol{c_1\cdot c_2}}\right); \tag{16}$$

for the double angle:

$$\mathbf{2}\boldsymbol{\alpha}=\left(\mathbf{1},\frac{\boldsymbol{2ab}}{\boldsymbol{c^2}},\frac{\boldsymbol{a+b^2}}{\boldsymbol{c^2}}\right); \tag{17}$$

for the half angle:

$$\frac{\boldsymbol{\alpha}}{\mathbf{2}}=\left(\mathbf{1},\sqrt{\frac{\boldsymbol{a-c}}{\boldsymbol{2c}}},\sqrt{\frac{\boldsymbol{a+c}}{\boldsymbol{2c}}}\right). \tag{18}$$

IV. THE APPLICATION OF THE RESULTS

Among many applications of trigonometry, the one that was already mentioned can be found - the application for the calculation of values when decomposing vectors. Since this is the example that prompted us to do this research - we will focus on it in the form of a steep plane.

The steep plane is widely used in experiments and theory, as well as in everyday life. One of the practical examples of using a steep plane in everyday life is a *ramp*. Ramps for people with disabilities are used by the people instead of stairs when they are unable to use them. Currently, stationary, fold-ing and sliding ramps are mostly in use.

When we look at the prescribed dimensions of the stairs [5], we notice that ramps can be made for all types of stairs using Pythagorean triples, not angles. Figure 5 shows possible angles

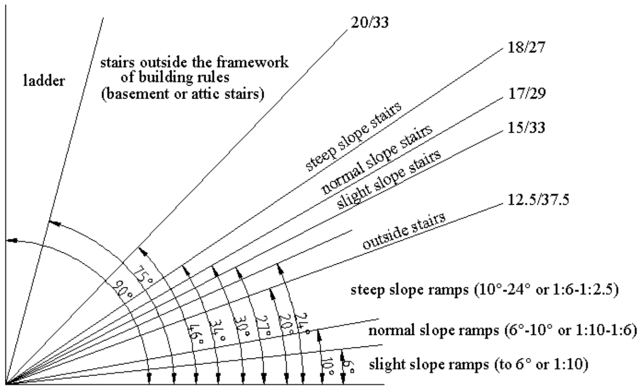

Figure 5. Possible angles of ramps, steps and ladders

for ramps, stairs and ladders.

When we look at the dimensions of the steps, we notice that ramps can be made for all types of stairs using Pythagorean triplets, not angles.

For steep slope stairs, where the height of the steps is 18 cm and their width is 27 cm, we can use the following dimensions of the ramps depending on the number of the steps:

1) 18 cm height, 80 cm width, 82 cm real length of the ramp,12.68° - steep slope ramp;

2) 36 cm height, 323 cm width, 325 cm real length of the ramp., 6.36° - normal slope ramp;

3) 54 cm height, 728 cm width, 730 cm real length of the ramp, 4.24° - slight slope ramp;

4) 72 cm height, 1295 cm width, 1297 cm real length of the ramp, 3.18 ° - slight slope ramp.

For the case when the steps are of normal slope, and the height of the steps is 17 cm and their length is 29 cm, we can use the following dimensions of the ramps depending on the number of steps:

1) 17 cm height, 144 cm width, 145 cm real lenght of the ramp, 6.73° - normal slope ramp;

2) 34 cm height, 288 cm width, 290 cm real lenght of the ramp, 6.73° - normal slope ramp;

3) 51 cm height, 1300 cm width, 1301 cm real lenght of the ramp, 2.25° - slight slope ramp;

4) 68 cm height, 1155 cm width, 1157 cm real lenght of the ramp, 3.37° - slight slope ramp.

Among many applications of hyperbolic functions, one can find their use in numerous formulas, such as those for hyperboloids. Hyperbolic and trigonometric functions also appear in the formulas for obtaining their coordinates. Single-layer hyperboloids are used in architecture and construction, with structures called hyperboloid structures. A hyperboloid can be built with straight steel beams, and as a product, a strong structure and more economical construction is obtained than with other methods. Examples include cooling towers,

 ISBN 978-86-7466-969-3

especially power plants, and many other structures. Apart from practical application, they can also appear for the sake of aesthetics, as famous Gaudi did for one of the most famous buildings "La Sagrada Familia" [15]. They can be seen in countless places in his buildings, for example as parts of roofs and staircases. Gaudí's circular windows are often shaped like a hyperboloid, rather than the more conventional cylinder, to improve lighting.

CONCLUSION

This approach to trigonometry and hyperbolic functions could contribute to a significant shortening of the procedures for calculating angle functions, which would find its application in other sciences and fields (computer science, astrono-my, physics, architecture, civil engineering (construction), etc.).

Guided by the idea that length is a basic physical quantity and that in practice it is possible to measure it more precisely, we conclude that in this way the construction of ramps on the ground is simplified (because data on the slope would be replaced by data on the length of the hypotenuse), as well as to enable a wider application of complex models that would simplify the construction of individual buildings.

ACKNOWLEDGMENT

The second author was partially financially supported by the Ministry of Science, Technological Development and Innovation of The Republic of Serbia, contract 451-03-47/ 2023-01/200103.

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