Determination of improved initial solution for approximation of arbitrary shape all-pass filter phase

Goran Stančić and Ivana Kostić

***Abstract*—A new approach to determine improved initial solution for all-pass phase approximation, is presented in this paper. Most filter design methods are reduced to iterative procedures that require a good initial solution to ensure stable convergence to the global minimum i.e. optimal solution. Therefore, choosing an adequate initial solution is extremely important. The proposed algorithm yields a solution to a variety of filters designed by implementation of IIR all-pass filters. Finally, examples have been provided to illustrate the effectiveness of this approach. The proposed method does not demand calculation of any derivative of objective function. The suggested algorithm secures quasi-equiripple phase error in a very few steps regardless the phase shape.**

***Index Terms*—Initial solution, all-pass filter, parallel structure, phase approximation, equiripple phase error, arbitrary phase.**

I. Introduction

Most of filters are designed to achieve a specified amplitude characteristic. This causes the deviation of the obtained phase response from the ideal one. Problem can be resolved by cascading of all-pass filters that work as a phase or group delay correctors. On the other hand, filters can be designed to approximate a certain phase characteristic. Compared to several traditional filter structures, parallel all-pass structure is more efficient for several reasons. The phase approximation is applicable to any all-pass recursive filter design problem. Filters designed according to this principle are suitable for real-time implementation due to parallel processing which provides less time delay for the same order of resulting filter.

There are two principles in design of all-pass filters based on phase approximation. The first one relays on the determination of the module and argument of the transfer function poles. The second order section could have either complex or real poles. The real poles may be positioned on positive and/or negative part of real axis inside the unit circle. Existence of positive real pole causes the first phase error extremum to be minimum, otherwise it would be maximum. According to that, the other approach has advantage due to unique optimal solution obtained in process of determining the filter coefficients. From initial solution the nature of the first extremum would be determined.

The problem of designing digital all-pass filters is still in the center of attention nowadays. The required phase shape of the all-pass filter depends on the type of the designed filter. Algorithms for determining the coefficients of the transfer function of these networks are mostly iterative and require knowledge of the initial solution, i.e. initial approximations. The problem is particularly observable when we deal with more complex networks, i.e. higher order system, when the iterative procedure converges to the global optimum only when the initial solution is sufficiently close to the final solution. The proposed method for initial solution determination can be equally well used in algorithms based on phase approximation and in group delay approximation procedures. For a given order of the filter, the approximation error curve needs to possess a maximum number of extrema, to allow this initial solution be used for the mini-max approximation.

The determination of good initial solution for approximation of desired group delay of all-pass filter is proposed in [1]. A method is iterative and use the simplified equation for group delay of all-pass filter, neglecting the influence of the poles from lower half of the unit circle which corresponds to negative frequencies. This method adapts each pole location sequentially, one at a time. The problem of designing recursive digital filters whose frequency response approximates an arbitrarily prescribed function in the Chebyshev sense on a single interval is considered in [2]. The minimum phase error response was used as approximation criterion to design the all-pass filters [3]. The shortages of this method include the needs of initial solutions and demanding computations. In [4], Tseng used the least-square phase error criterion to design digital IIR all-pass filters to have an arbitrarily predefined phase response. The method is used to design an unconstrained all-pass filter whose optimal filter coefficients are obtained by solving a set of linear equations in each iteration.

Digital filter's realization can be based on two all-pass filters in parallel configuration structure [5]. In the parallel branches there are filters of the appropriate order. Direct path in one branch leads to the constant phase while delay line would provide selective filter with approximately linear phase in design of piece-wise constant magnitude [6].

Different filters such as band-stop, band-pass, lowpass and highpass filters can be realized with this approach. Furthermore, this configuration can achieve all the given specifications of the digital differentiator. It is quite well

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known, that these realizations provide low passband sensitivity. The full advantage of this configuration is achieved when it is necessary to implement both complementary filters because the second one demands only one additional adder.

A frequently applied approximation is equiripple approximation of the desired phase or magnitude response. The final equiripple solution is reached by iterative procedures that require an initial solution. The iterative procedure belongs to the class of mini-max approximation algorithms. The error function's extrema of initial solution could have significantly different absolute values across the full frequency range. This is particularly expressive for high-order filters. Very big difference between values of the phase error extrema introduce problem how to correctly choose the allowed maximum phase deviation for first iteration. Poorly adopted permitted phase error cause more divergence. To mitigate the problem, differences between extrema of the phase error function should be minimal. The values of extrema of the phase error get closer to each other in absolute value, and the solution gradually becomes a quasi-equiripple phase error digital filter. This is achieved with the proposed iterative procedure. It is generally known that a higher order filter provides a smaller approximation error. The frequencies of digital filters are in the range [0, π]. By increasing the order of the filter, the area covered by one pole decreases. The idea how to reach equripple phase characteristic is based on assimilation of extrema. Intuitively we expect that phase error extrema values decrease if the dedicated approximation region is narrowed. On the contrary, if an approximation region becomes wider, the value of corresponding extremum increases.

Although the phase approximation algorithm may find other applications, here in the present work we only discuss its application to the all-pass digital filter design problem. In this paper we introduce a new method for designing digital all-pass filters. According to a given desired phase response, a set of all-pass digital filter coefficients is to be found such that the maximum phase error is minimized. The proposed method ensures an initial solution that would offer enough equations to determine all transfer function coefficients of the N-th order all-pass filter. The proposed approach can be applied to arbitrary phase shape.

II. Initial Solution Determination

The IIR all-pass filter of order *N* follows the response

$$H_N(z) = \frac{a_N + a_{N-1}z^{-1} + \cdots + a_1 z^{-N+1} + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \tag{1}$$

By substituting $z = e^{j\omega}$ into (1) and after simple mathematical manipulations the phase characteristic of the digital all-pass filter can be obtained as:

$$\varphi_N(\omega) = -N\omega + 2\arctan\frac{\sum_{i=1}^{N} a_i \sin(i\omega)}{1 + \sum_{i=1}^{N} a_i \cos(i\omega)} \tag{2}$$

The relations are derived according to which it is possible to obtain the unknown transfer function coefficients of all-pass filter. Determination of improved initial solution is specified for achievement of constant, linear and quadratic phase.

A. Design of differentiator

The differentiator design can be based on two IIR all-pass filters in parallel structure, with their orders satisfying N = M + 1. To obtain filter with magnitude response to be linear function of frequency the difference between phases of the all-pass filters has to be specific [7]. One parallel branch could be either direct path or delay line. If delay line of order M is applied, all-pass filter phase needs to approximate ideal phase of the form

$$\varphi_{Ideal_N}(\omega) = -M\omega - 2\arcsin\left(\omega/\pi\right) \tag{3}$$

The phase error curve has N+1 extrema. Hence, there are N frequencies ω_i at which the phase error must have a zero-value defined as

$$\omega_i = \frac{\pi i}{N+1}, \quad i = 0, 1, \ldots, N+1 \tag{4}$$

For $i = 0$ and $i = N+1$ the phase error is surely zero, taking into account the all-pass filter phase properties. The zone's width zw is the area at the frequency axis between two phase error zeros. Inside the zone, phase error has the same sign. So, the interval [0, π] is splitted into N +1 zones of the equal width:

$$zw_i = \omega_i - \omega_{i-1}, \quad i = 1, \ldots, N+1 \tag{5}$$

The proposed method is based on the zone's width modification in order to achieve all extrema to have similar values. In every iteration step it is necessary to define the area reserved for correction of all zones. The total correction of zone's width in *k*-th iteration is given with parameter Δ which is experimentally obtained as

$$\Delta = \frac{\pi}{2^k D}, \quad D = \begin{cases} 4N, & N = 2 \\ 3N, & N = 3 \\ 2N, & N \geq 4 \end{cases} \tag{6}$$

The absolute values of phase error extrema ε_i in all zones are determined as

$$\varepsilon_i = \max\left|\varphi_N(\omega) - \varphi_{Ideal_N}(\omega)\right|, \quad \omega \in zw_i, \quad i = 1, \ldots, N+1 \tag{7}$$

In *k*-th iteration, the average value of phase error extrema is calculated as

$$\varepsilon_k = \frac{1}{N+1}\sum_{i=1}^{N+1} \varepsilon_i \tag{8}$$

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Deviations δ_i of the phase error extrema ε_i from the average value ε_k are $\delta_i = \varepsilon_i - \varepsilon_k$, $i = 1, \ldots, N+1$. In the next step, sum of absolute values of all maxima deviation from the average value need to be determined

$$\delta_{MAX} = \sum_{j=1}^{N_{MAX}} \left|\delta(2j-p)\right| \tag{9}$$

Similarly, the total deviation of all minima can be formulated as

$$\delta_{MIN} = \sum_{j=1}^{N_{MIN}} \left|\delta(2j-1+p)\right| \tag{10}$$

where p is a parameter, whose value depends on the parity of the all-pass filter order, given by

$$p = \begin{cases} 0, & \text{even } N \\ 1, & \text{odd } N \end{cases} \tag{11}$$

Also, $N_{MAX} + N_{MIN} = N+1$ holds. The new corrected values of zone's widths $\overline{zw_i}$ are obtained by

$$\overline{zw_i} = zw_i - \frac{\Delta \cdot \delta_i}{\delta_{MAX}} \qquad \overline{zw_i} = zw_i - \frac{\Delta \cdot \delta_i}{\delta_{MIN}} \tag{12}$$

which correspond to the zones with positive and negative phase error, respectively. Practically, the zones are modified according to their extremum influence on total deviation of the maxima i.e. minima. The new locations of the phase error zero frequencies are given by

$$\overline{w_i} = \overline{w_{i-1}} + \overline{zw_i} \qquad \overline{w_0} = 0 \qquad i = 1, 2, \ldots N+1 \tag{13}$$

The counter k is increased and the values of new phase error zeros and zone's width to the current variables are assigned.

$$k \leftarrow k+1 \quad \omega_i \leftarrow \overline{\omega_i} \quad zw_i \leftarrow \overline{zw_i} \tag{14}$$

At this point one iterative step is finished and one can proceed to the next step after halving Δ according to equation (6). Every beam in Fig. 1 and Fig. 2 corresponds to one zone where phase error has the same sign. From the phase error, presented in Fig. 1 for all-pass filter of order N=6, it is evident that the initial solution has the sufficient number of extrema but deviation between them is considerably large.

Zone's width is displayed in Fig. 2. The last zone given with brown bar has the biggest extremum and according to given procedure is downsized to become narrowest in order to equalize all extrema. For all-pass filter N=10 with $m_1 = 3$, $m_2 = 7$ zone's widths are displayed in Fig 3.

As evident, the obtained phase error extrema have similar values so correction of zone's width will be minor when the proposed procedure for error equalization will be applied.

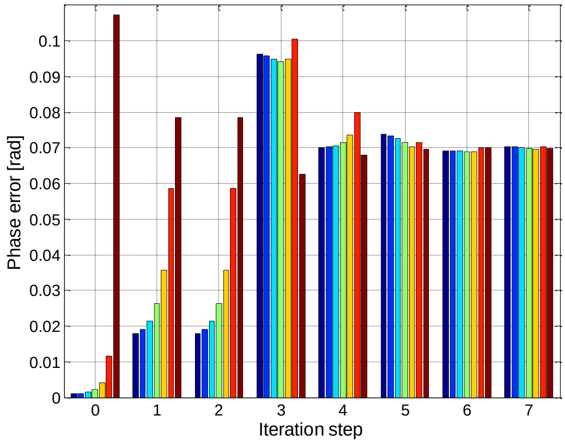

Fig. 1. Values of all-pass filter of order N=6 phase error extrema for the initial solution and the first seven iterations obtained by proposed procedure in design of differentiator

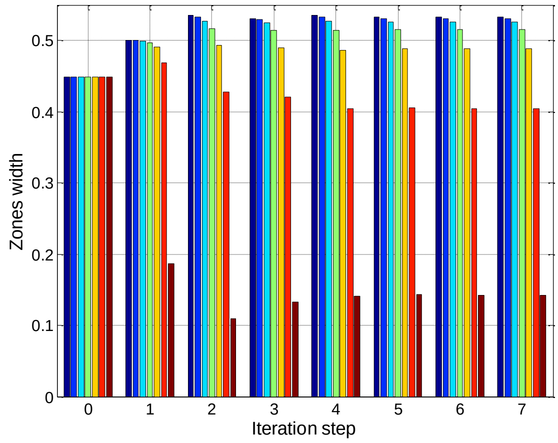

Fig. 2. Values of zone's widths of all-pass filter of order N=6 obtained in the first seven iterations starting with equidistant phase error zeros i.e. equal zones

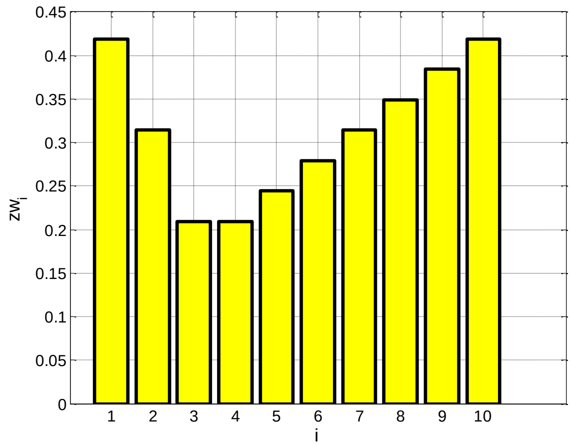

Fig. 3. Zone's widths of initial solution for all-pass filter of order N=10 for R=2

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As the algorithm iterates, the extrema get closer to each other in absolute value. Only four iterations were sufficient to get all-pass filter with quasi-equiripple error. At that point it is easy to choose allowed phase error for further equiripple design as the mean value of all extrema taken with absolute values.

The numerous experiments have shown that described procedure is powerful but demand carefully adopted parameter Δ. Adequately chosen Δ allows to successfully reach the equiripple solution with no need for further correction and derivate calculation. The criterion for termination of proposed procedure in that case is point where temporarily Δ value is less than adopted arbitrarily small number.

B. Design of lowpass and bandpass filters

The selective IIR filters with piece-wise constant magnitude could be realized as parallel connection of two all-pass filters. The direct path in one branch leads to the resulting filter with approximately constant phase. The order of other all-pass filter depends on the number of bands of selective filter.

In the case of a lowpass/highpass filter design an all-pass filter has to be of the first order for instance. Such approach does not offer the opportunity to choose more complex all-pass filter in order to obtain lower approximation error. For instance, the second order all-pass filter is used if the aim is a notch or a band-pass/band-stop filter. Such limitations do not exist if the delay line is in one branch. Resulting filter would have the approximately linear phase and the approximation error is directly influenced by the all-pass filter's order N. The order of delay line M directly depends on resulting filter nature. For instance, to obtain lowpass/highpass filter, it is necessary to holds N=M+1. To realize band-stop/band-pass filter N=M+2 holds. Number of local extrema on the phase error characteristics in the passband is m_1. The total number of extrema is N and m_2 is number of extrema in the stopband.

In every iteration step the all-pass filter phase approximates ideal linear phase given by

$$\varphi(\omega)=\begin{cases}-M\omega & 0<\omega<\omega_p\\ -M\omega-\pi & \omega_s<\omega<\pi\end{cases} \tag{15}$$

As in differentiator case, the initial solution will be obtained solving system of equations

$$\varphi(\omega_i)-\varphi_N(\omega_i)=0 \quad i=1,2,\dots N \tag{16}$$

where are points on the frequency axis, where the phase error is equal to zero. Starting from equation (2), equation (17) can be given, after mathematical manipulations, in matrix form as

$$\begin{bmatrix}\sin(\omega_1)-\tan(f(\omega_1))\cos(\omega_1) & \cdots & \sin(N\omega_1)-\tan(f(\omega_1))\cos(N\omega_1)\\ \sin(\omega_2)-\tan(f(\omega_2))\cos(\omega_2) & \cdots & \sin(N\omega_2)-\tan(f(\omega_2))\cos(N\omega_2)\\ \vdots & \ddots & \vdots\\ \sin(\omega_N)-\tan(f(\omega_N))\cos(\omega_N) & \cdots & \sin(N\omega_N)-\tan(f(\omega_N))\cos(N\omega_N)\end{bmatrix}\cdot \begin{bmatrix}a_1\\ a_1\\ \vdots\\ a_N\end{bmatrix}=\begin{bmatrix}\tan(f(\omega_1))\\ \tan(f(\omega_2))\\ \vdots\\ \tan(f(\omega_N))\end{bmatrix} \tag{17}$$

where

$$f(\omega)=\frac{\omega}{2}-\frac{k\pi}{2} \tag{18}$$

which solving gives the transfer function coefficients a_i. The interval [0, π] is now splitted into N zones.

$$zw_i=\omega_i-\omega_{i-1},\quad i=1,\dots,N \tag{19}$$

To remind, in the differentiator case we start with all equal zones and consequence was significant difference between extrema values. After seven iterations the extrema become similar but the last zone is three times narrower than the others. At high frequencies all-pass filter phase deviate the most from ideal one. In order to get more uniform values for the phase error extrema of starting solution we decide to abandon the equal zone's width at the beginning.

The phase jump of π radians in the transition zone of lowpass filter will provide a pole the closest to the unit circle and it demands the narrowest zone. One can choose the ratio R between the widest and the narrowest zone. To achieve that, the normalized zone's width in the passband is

$$nzw_i=1-(R-1)\cdot\frac{i-m_1}{m_1-1},\quad i=1,\dots,m_1 \tag{20}$$

and in the stopband are

$$nzw_i=R+(R-1)\cdot\frac{i-N-1}{m_2-1},\quad i=m_1+1,\dots,N+1 \tag{21}$$

Zone's width is obtained from

$$zw_i=\frac{nzw_i\cdot\pi}{\sum nzw_i} \tag{22}$$

Also, the phase and phase error of the initial solution for all-pass filters N=10 and N=6 are given in Fig. 4. The phase response curves of these filters have 10 i.e. 6 extrema, respectively. For both all-pass filters, in the passband there are 3 extrema and the rest of them are located in the stopband. Further equalization of extrema values could be performed by applying the proposed procedure of zones width modification.

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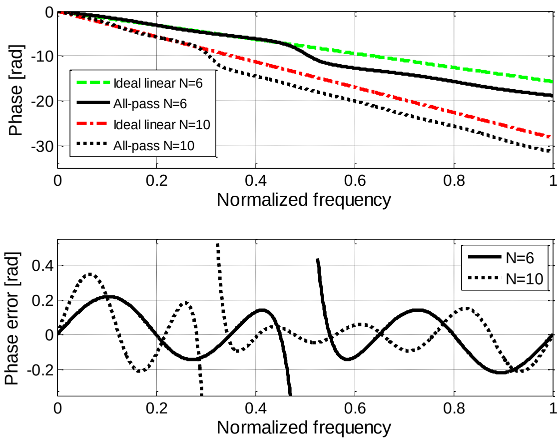

Fig. 4. The ideal phase, phase and phase error of the initial solutions for all-pass filters of order N=6 and N=10

Now, parameter Δ, defined as the total area dedicated for correction of zone's width in the first iteration, has value

$$\Delta = \frac{\pi}{N+C} \qquad 0 < C \le 1 \tag{23}$$

The numerous experiments have shown that good choice of parameter C is a value near unity for low order filters and decreased values toward zero for high order filters.

Through iterations the absolute values of phase error extrema ε_i in all zones are determined and correction is performed according to earlier described procedure. The results show that already after four iterations, the extrema on the phase error curve are approximately equal which ensures quasi-equirrple phase error characteristic. The main goal of the initial solution is to ensure an adequate number of phase error extrema. Obtained results for filter of order N=10 are displayed in Figs. 5 and 7, and for filter of order N=6 are displayed in Figs. 6 and 8. Good choice of starting zone's width values helps to reach quasi-equiripple phase error in only four iterations. The values of maximum phase error in all zones are displayed in Figs. 7 and 8 for all-pass filters of order N=10 and N=6, respectively.

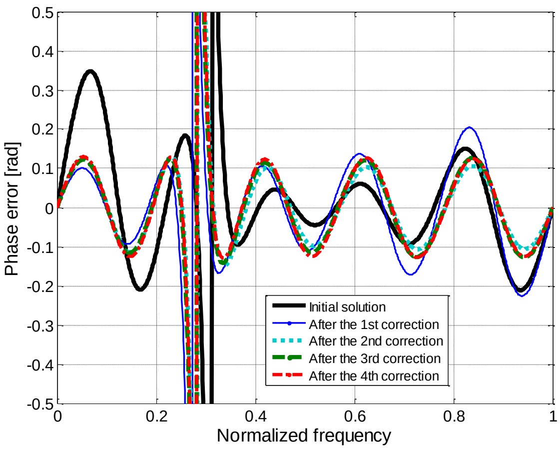

Fig. 5. Phase error of the all-pass filter of order N=10 for the first four iterations

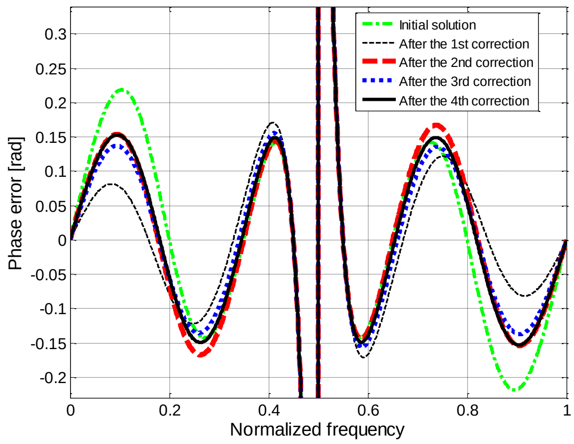

Fig. 6. Phase error of the all-pass filter of order N=6 for the first four iterations

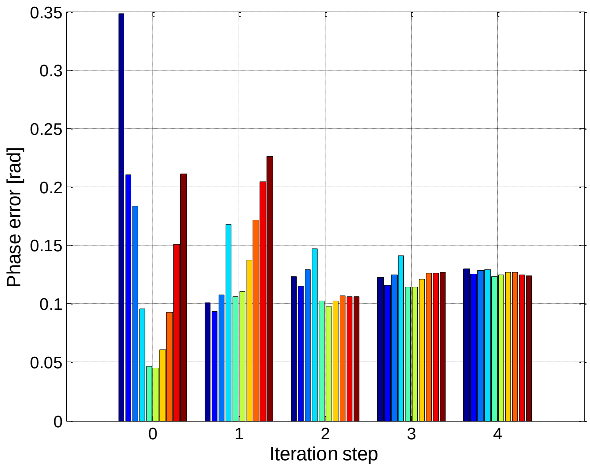

Fig. 7. Values of the phase error extrema of the all-pass filter of order N=10 for the initial solution and first four iterations

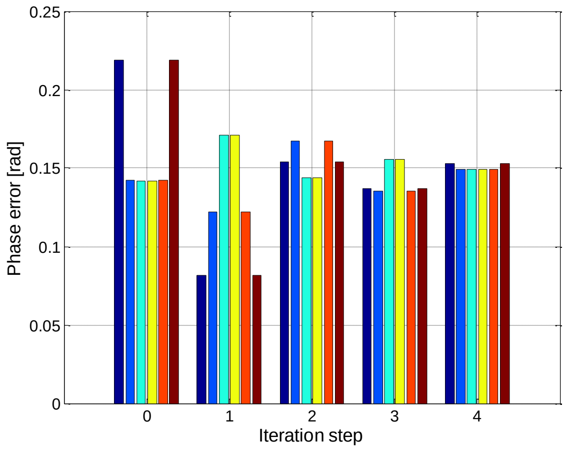

Fig. 8. Values of the phase error extrema of the all-pass filter of order N=6 for the initial solution and first four iterations

It is important to emphasize that in some filter design (for specific all-pass filter phase shape), it could happen that solving system of equations (17) with equidistant frequency points where the phase error have zero value, does not give stable filter. That was the reason to introduce unequal zone's width.

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C. Design of filter with quadratic phase characteristic

Linear group delay i.e. the quadratic phase filters are used for expansion or compression of chirp signals in radar systems [8]. The area below the group delay of all-pass filter of second order is exactly 2π. If group delay is linear function of frequency one can expect the denser poles distribution in area with higher group delay. The normalized zone's width is defined as (24) where R gives the ratio of the widest and narrowest zone.

$$nzw_i = R + (R-1)\cdot\frac{i-N-1}{N},\quad i=1,\ldots,N+1 \qquad (24)$$

The zone's width could be obtained using equation (23). The impact of the assigned correction area in the first few iterations is shown in Fig. 9. Experiments have shown that choice $\Delta = \pi/N$ is appropriate regardless the filters order. The four iterations were enough to reach the good starting point for equiripple design. In the given example quadratic phase $A\omega^2 + B\omega$ was approximated where $A = -1/\pi$ and $B = -N - A\cdot\pi$.

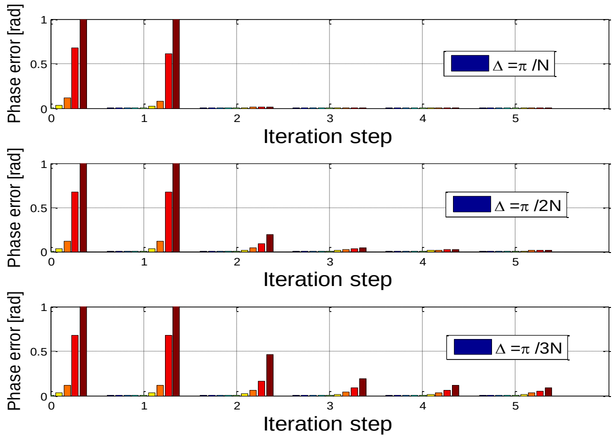

Fig. 9. The impact of the assigned correction area on values of the phase error extrema for all-pass filter of order N=8 for the initial solution and first five iterations

III. Conclusion

We provide in this paper a procedure for determining of an improved initial solution for all-pass filter design. The vector of determined coefficients of the all-pass filter provides an appropriate number of the phase error extrema. The analyzed methodology ensures stable convergence and is not resctricted by the filter type. It is shown that constant, linear or quadratic phase characteristic could be achieved with the proposed iterative procedure.

Numerous experiments of differentiators, lowpass filters and filters for pulse compression design confirmed the effectiveness of the described procedure. The speed of convergence toward Approach of determining to the quasi-equiripple phase error solution is influenced by adopted area dedicated to the zone's width correction.

In general, the approach is based on the fact that expanding the width of the zone will cause the corresponding error extremum to increase. Similarly, shrinking of the zone will cause the phase error extremum to decrease. Hence, with the proposed method, a quasi-equirriple phase error can be obtained in a relatively small number of iterations without the need to determine the derivative of the objective function.

The described procedure is efficient and can be used as starting point in design of high order filters. If at the very start an adequate value of area reserved for correction of the zone's width is adopted, the described procedure successfully leads to equripple phase filter with no need for implementation of other methods.

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