# Current Distribution in an Inhomogeneous Conductor in the Presence of a Filament 

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#### Abstract

In this paper we present a separate rigorous analysis of the skin and proximity effects in an inhomogeneous conductor in the presence of a filament. The conductor consists of a massive central conductor coated with a thin layer of different conductivity. The skin and proximity solutions are assumed in the form of two infinite sums of the proper harmonics. The unknown coefficients in the skin effect case are found by applying boundary conditions, while a system of two integral equations is used to determine the unknown coefficients in the proximity effect solution, with no boundary conditions involved.


Index Terms-skin effect, proximity effect, inhomogeneous conductor, filament, current distribution, integral equation

## I. Introduction

There are very few cases where a solution for the current distribution of time-varying currents can be obtained in a closed form. These cases include some conductor configurations in the presence of a filament - massive circular conductor [1-3], thin tubular conductor [4,5], hollow massive conductor [2,6] and thin two-layer tubular conductor [7]. The method of integral equations proved to be very powerful in these cases, requiring no boundary conditions when the conductor is homogeneous.

In this paper we investigate the skin and proximity effects in the case of a massive circular conductor covered by a thin layer, in the presence of a filament. Although the conductor is inhomogeneous, no boundary conditions are required when analyzing the proximity part of the problem.

## II. FORMULATION OF THE PROBLEM AND GENERAL FORM OF SOLUTION

Geometry of the problem is shown in Fig. 1. An inhomogeneous conductor consisting of a massive conductor of radius $a$ and conductivity $\sigma_{1}$ coated with a thin layer of thickness $d(d \ll a)$, and conductivity $\sigma_{2}$, and a filament carry equal opposite sinusoidal currents of rms $I$ and frequency $f$. Distance between the conductor axis and the filament is $D$. The object is to find current distribution in the conductor.

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Fig. 1 Inhomogeneous conductor and filament

The proper radial harmonics in cylindrical coordinates are modified Bessel functions $I_{n}(k r)$ and $K_{n}(k r)\left(k^{2}=j \omega \mu_{0} \sigma\right)$, and the proper angular harmonics are trigonometrical functions $\cos n \theta$ and $\sin n \theta$. Due to symmetry, the sine function must be excluded since the current density should be an even function of $\theta$. In region 1, the $K_{n}$ function should not be present since it is unbounded for $r=0$. Hence, the general form of the solution for current density in region 1 is

$$
\begin{equation*}
J_{1}(r, \theta)=\sum_{n=0}^{\infty} A_{n} I_{n}\left(k_{1} r\right) \cos n \theta, k_{1}^{2}=j \omega \mu_{0} \sigma_{1} \tag{1}
\end{equation*}
$$

In region 2, due to its small thickness, the radial dependence may be neglected, so that the general form of the solution in this region is

$$
\begin{equation*}
J_{2}(\theta)=\sum_{n=0}^{\infty} B_{n} \cos n \theta \tag{2}
\end{equation*}
$$

In (1) - (2), $A_{n}$ and $B_{n}$ are unknown coefficients that should be determined.

It is convenient to treat separately the skin and proximity effects in this problem. The skin-effect solutions are represented by the first terms $(n=0)$ in (1) - $(2)$, while the remaining infinite sums ( $n \geq 1$ ) account for proximity effect.

### 2.1 Skin-effect solution

As mentioned above, this solution is given by

$$
\begin{equation*}
J_{1}^{s}(r)=A_{0} I_{0}\left(k_{1} r\right) \tag{3}
\end{equation*}
$$

$$
\begin{equation*}
J_{2}^{s}(r)=B_{0}=\text { const } . \tag{4}
\end{equation*}
$$

Equality of the tangential components of the electrical fields at the interface $r=a$ requires that

$$
\frac{A_{0}}{\sigma_{1}} I_{0}\left(k_{1} r\right)=\frac{B_{0}}{\sigma_{2}}
$$

or

$$
\begin{equation*}
A_{0} \chi^{2} I_{0}\left(k_{1} r\right)=B_{0} \tag{5}
\end{equation*}
$$

where

$$
\chi^{2}=\frac{\sigma_{2}}{\sigma_{1}}=\frac{k_{2}^{2}}{k_{1}^{2}}
$$

The total current through the conductor is

$$
I=\int_{S_{1}} J_{1}(r, \theta) r d r d \theta+a d \int_{0}^{2 \pi} J_{2}(\theta) d \theta
$$

where only the first terms $(n=0)$ from (1) $-(2)$ should be taken, since $\int_{0}^{2 \pi} \cos n \theta d \theta=0, n \geq 1$. After integration,

$$
\begin{equation*}
I=\frac{2 \pi}{k_{1}} A_{0} I_{1}\left(k_{1} a\right)+2 \pi a d B_{0} \tag{6}
\end{equation*}
$$

From (5) - (6) we can find the unknown coefficients

$$
\begin{align*}
& A_{0}=\frac{I k_{1}^{2}}{2 \pi a} \frac{1}{k_{1} I_{1}\left(k_{1} a\right)+k_{2}^{2} d I_{0}\left(k_{1} a\right)}  \tag{7}\\
& B_{0}=\frac{I k_{2}^{2}}{2 \pi a} \frac{I_{0}\left(k_{1} a\right)}{k_{1} I_{1}\left(k_{1} a\right)+k_{2}^{2} d I_{0}\left(k_{1} a\right)} \tag{8}
\end{align*}
$$

and the skin- effect solution given by (3) - (4) and (7) - (8) is completed.

### 2.2. Proximity-effect solution

To find the unknown coefficients $A_{n}$ and $B_{n}(n \geq 1)$ in the proximity effect solutions

$$
\begin{gather*}
J_{1}^{p}(r, \theta)=\sum_{n=0}^{\infty} A_{n} I_{n}\left(k_{1} r\right) \cos n \theta  \tag{9}\\
J_{2}^{p}(r, \theta)=\sum_{n=0}^{\infty} B_{n} \cos n \theta \tag{10}
\end{gather*}
$$

we will use the method of integral equations. These equations for the case of two conductors (two regions of the
inhomogeneous conductor) in the presence of a filament have the following form in cylindrical coordinates [8].

$$
\begin{align*}
& J_{1}(r, \theta)=\frac{k_{1}^{2}}{4 \pi}\left[\int_{S_{1}} J_{1}\left(r^{\prime}, \theta^{\prime}\right) \ln f\left(r, r^{\prime}, \theta, \theta^{\prime}\right) d S_{1}+\right. \\
& +\int_{S_{2}} J_{2}\left(r^{\prime}, \theta^{\prime}\right) \ln f\left(r, r^{\prime}, \theta, \theta^{\prime}\right) d S_{2}-  \tag{11}\\
& -I \ln f(r, D, \theta, 0)]+k_{1}, \quad r \in S_{1}
\end{align*}
$$

$$
\begin{align*}
& J_{2}(r, \theta)=\frac{k_{2}^{2}}{4 \pi}\left[\int_{S_{1}} J_{1}\left(r^{\prime}, \theta^{\prime}\right) \ln f\left(r, r^{\prime}, \theta, \theta^{\prime}\right) d S_{1}+\right. \\
& +\int_{S_{2}} J_{2}\left(r^{\prime}, \theta^{\prime}\right) \ln f\left(r, r^{\prime}, \theta, \theta^{\prime}\right) d S_{2}-  \tag{12}\\
& -I \ln f(r, D, \theta, 0)]+K_{2}, \quad r \in S_{2}
\end{align*}
$$

where

$$
f\left(r, r^{\prime}, \theta, \theta^{\prime}\right)=\frac{r^{2}+r^{\prime 2}-2 r r^{\prime} \cos \left(\theta-\theta^{\prime}\right)}{D^{2}}
$$

In (11) - (12) $K_{1}$ and $K_{2}$ are some unknown constants. For our specific case we may take in (11)-(12) $r^{\prime} \approx a, d S_{2}=d a d \theta$ in the integrals over $S_{2}$. We may also take $r \approx a$ in (12), and (11) - (12) become

$$
\begin{align*}
J_{1}(r, \theta)= & \frac{k_{1}^{2}}{4 \pi}\left[\int_{S_{1}} J_{1}\left(r^{\prime}, \theta^{\prime}\right) \ln f\left(r, r^{\prime}, \theta, \theta^{\prime}\right) d S_{1}+\right. \\
& +\int_{S_{2}} J_{2}\left(\theta^{\prime}\right) \ln f\left(r, a, \theta, \theta^{\prime}\right) d a d \theta-  \tag{13}\\
& -I \ln f(r, D, \theta, 0)]
\end{align*}
$$

$$
\begin{align*}
& J_{2}(r, \theta)=\frac{k_{2}^{2}}{4 \pi}\left[\int_{S_{1}} J_{1}\left(r^{\prime}, \theta^{\prime}\right) \ln f\left(r, r^{\prime}, \theta, \theta^{\prime}\right) d S_{1}+\right. \\
& +\int_{S_{2}} J_{2}\left(\theta^{\prime}\right) \ln f\left(r, a, \theta, \theta^{\prime}\right) d a d \theta-  \tag{14}\\
& -I \ln f(a, D, \theta, 0)]+K_{2}, r \in S_{2}
\end{align*}
$$

Now, we substitute (9) - (10) into (13) - (14) to get

$$
\begin{align*}
& \sum_{n=0}^{\infty} A_{n} I_{n}\left(k_{1} r\right) \cos n \theta=\frac{k_{1}^{2}}{4 \pi}\left[\sum_{n=1}^{\infty} A_{n} P_{n}(r, \theta)+\right.  \tag{15}\\
& +a d \sum_{n=1}^{\infty} B_{n} Q_{n}(r, \theta)+2 I \sum_{n=1}^{\infty}\left(\frac{r}{D}\right)^{n} \frac{\cos n \theta}{n}+K_{1}
\end{align*}
$$

$$
\begin{align*}
& \sum_{n=0}^{\infty} B_{n} \cos n \theta=\frac{k_{2}^{2}}{4 \pi}\left[\sum_{n=1}^{\infty} A_{n} R_{n}(\theta)+\right.  \tag{16}\\
& +a d \sum_{n=1}^{\infty} B_{n} S_{n}(\theta)+2 I \sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^{n} \frac{\cos n \theta}{n}+K_{2}
\end{align*}
$$

where we also used the expansion

$$
\begin{equation*}
\ln \left(1+x^{2}-2 x \cos \theta\right)=-2 \sum_{n=1}^{\infty} x^{n} \frac{\cos n \theta}{n,},|x|<1 \tag{17}
\end{equation*}
$$

In (15) - (16), $P_{n}, Q_{n}, R_{n}$ and $S_{\mathrm{n}}$ are integrals given by

$$
\begin{gather*}
P_{n}(r, \theta)=\int_{0}^{2 \pi} \cos n \theta^{\prime} d \theta^{\prime} \int_{0}^{a} r^{\prime} I_{n}\left(k_{1} r^{\prime}\right) \ln \frac{r^{2}+r^{\prime 2}-2 r r^{\prime} \cos \left(\theta-\theta^{\prime}\right)}{D^{2}} d r^{\prime}  \tag{18}\\
Q_{n}(r, \theta)=\int_{0}^{2 \pi} \cos n \theta^{\prime} d \theta^{\prime} \ln \frac{r^{2}+a^{2}-2 r a \cos \left(\theta-\theta^{\prime}\right)}{D^{2}} d \theta^{\prime}  \tag{19}\\
R_{n}(\theta)=P_{n}(a, \theta)  \tag{20}\\
S_{n}(\theta)=Q_{n}(a, \theta) \tag{21}
\end{gather*}
$$

Integral (18), with $I_{n}\left(k_{1} r^{\prime}\right)$ replaced by $J_{n}(\mathrm{j} k r)$ (the Bessel function the first kind), was evaluated in [3]. Using the result and a simple interrelation between $J_{n}$ and $I_{n}$ we readily find

$$
\begin{equation*}
P_{n}(r, \theta)=-\frac{2 \pi a}{n k_{1}}\left(\frac{r}{a}\right)^{n} \cos n \theta I_{n-1}\left(k_{1} a\right)+\frac{4 \pi}{k_{1}^{2}} I_{n}\left(k_{1} r\right) \cos n \theta \tag{22}
\end{equation*}
$$

From (20) we find

$$
\begin{equation*}
R_{n}(\theta)=-\frac{2 \pi a}{n k_{1}} \cos n \theta I_{n+1}\left(k_{1} a\right) \tag{23}
\end{equation*}
$$

In deriving (23) we used the following formula [9]

$$
\begin{equation*}
I_{n-1}(z)-I_{n+1}(z)=\frac{2 n}{z} I_{n}(z) \tag{24}
\end{equation*}
$$

Integral (21) was evaluated in [7]. It is

$$
\begin{equation*}
S_{n}(\theta)=-\frac{2 \pi}{n} \cos n \theta \tag{25}
\end{equation*}
$$

The remaining integral given by (19) is evaluated in the Appendix. We here state the result

$$
\begin{equation*}
Q_{n}(r, \theta)=-\frac{2 \pi}{n}\left(\frac{r}{a}\right)^{n} \cos n \theta \tag{26}
\end{equation*}
$$

After substituting (22) - (23) and (25) - (26) into (15) - (16) we may equate the coefficients with $r^{n} \cos n \theta$ in (15), and the coefficients with $\cos n \theta$ in (16). This results in two equations

$$
\begin{equation*}
A_{n} I_{n-1}\left(k_{1} a\right)+k_{1} d B_{n}=\frac{I k_{1}}{\pi a}\left(\frac{a}{D}\right)^{n} \tag{27}
\end{equation*}
$$

$$
\begin{equation*}
A_{n} \frac{k_{1}^{2} a}{2 n k_{1}} I_{n+1}\left(k_{1} a\right)+B_{n}\left(1+\frac{k_{2}^{2} a d}{2 n}\right)=\frac{I k_{2}^{2}}{2 \pi}\left(\frac{a}{D}\right)^{n} \frac{1}{n} \tag{28}
\end{equation*}
$$

From (27) - (28), using also (24) we find the unknown coefficients

$$
\begin{align*}
& A_{n}=\frac{I k_{1}^{2}}{\pi a}\left(\frac{a}{D}\right)^{n} \frac{1}{k_{1} I_{n-1}\left(k_{1} a\right)+k_{2}^{2} d I_{n}\left(k_{1} a\right)}, \quad n \geq 1  \tag{29}\\
& B_{n}=\frac{I k_{2}^{2}}{\pi a}\left(\frac{a}{D}\right)^{n} \frac{I_{n}\left(k_{1} a\right)}{k_{1} I_{n-1}\left(k_{1} a\right)+k_{2}^{2} d I_{n}\left(k_{1} a\right)}, \quad n \geq 1 \tag{30}
\end{align*}
$$

Therefore, the proximity-effect solution given by (9) - (10) and (29) - (30) is now complete.

Finally, the total solution is obtained by summing the skin and proximity effect solutions.
It is worth to notice that no boundary conditions are used is solving the proximity effect problem. But it can be seen by a simple inspection from (29)-(30) that the boundary condition for the tangential electric fields at the interface $r=a$

$$
\frac{A_{n}}{\sigma_{1}} I_{n}\left(k_{1} a\right)=\frac{B_{n}}{\sigma_{2}}
$$

is satisfied.
From the obtained results some special cases immediately follow: solid conductor and filament [1]-[3] (we let $k_{2}=0$ in the expressions for $A_{n}, B_{n}, n \geq 0$ ), and thin tubular conductor and filament [4]-[5]. In the latter case we let $k_{1} \rightarrow 0$ and use the small argument approximation [9]

$$
I_{n}(z) \approx \frac{z^{n}}{2^{n} n!}, \quad|z| \ll 1
$$

## III. Conclusion

A rigorous analysis of the current distribution in a massive cylindrical conductor with a thin layer in the presence of a filament is presented in this paper. A solution is found in the two regions in the form of infinite sums of the proper harmonics, and the unknown coefficients are found from boundary conditions and from two integral equations.

## ApPENDIX

In this Appendix we evaluate integral (19)

$$
\begin{aligned}
Q_{n}(r, \theta) & =\int_{0}^{2 \pi} \cos n \theta^{\prime} \ln \frac{r^{2}+a^{2}-2 r a \cos \left(\theta-\theta^{\prime}\right)}{D^{2}} d \theta^{\prime}= \\
& =\int_{0}^{2 \pi} \cos n \theta^{\prime} \ln \left(\frac{a^{2}}{D^{2}} \frac{r^{2}+a^{2}-2 r a \cos \left(\theta-\theta^{\prime}\right)}{a^{2}}\right) d \theta^{\prime}= \\
& =\int_{0}^{2 \pi} \cos n \theta^{\prime} \ln \frac{r^{2}+a^{2}-2 r a \cos \left(\theta-\theta^{\prime}\right)}{a^{2}} d \theta^{\prime}
\end{aligned}
$$

using (17) with $x=\frac{r}{a}$ we can write

$$
Q_{n}(r, \theta)=-2 \int_{0}^{2 \pi} \cos n \theta^{\prime} \sum_{m=1}^{n}\left(\frac{r}{a}\right)^{m} \frac{\cos m\left(\theta-\theta^{\prime}\right)}{m} d \theta^{\prime}
$$

Only the term $m=n$ should be kept the infinite sum, due to orthogonality of the cosine function, hence

$$
\begin{aligned}
Q_{n}(r, \theta) & =-2 \cos n \theta\left(\frac{r}{a}\right)^{n} \frac{1}{n} \int_{0}^{2 \pi} \cos ^{2} n \theta^{\prime} d \theta^{\prime} \\
& =-\frac{2 \pi}{n}\left(\frac{r}{a}\right)^{n} \cos n \theta
\end{aligned}
$$

Note that $S_{n}(\theta)$, given by (25), can also be derived from (21) and (A1).

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