Current distribution in a hollow cylindrical conductor influenced by a parallel filament

Dragan Filipović, Tatijana Dlabač

***Abstract*—This paper presents a rigorous solution for the current distribution in a hollow cylindrical conductor in the presence of a current filament placed outside the conductor. The currents are assumed low-frequency, time-harmonic, flowing in opposite directions. As a starting point, we chose a Fredholm-type integral equation for the current density whose solution is sought in the form of an infinite sum of the proper harmonics – the modified Bessel functions of the second kind and trigonometric functions. The unknown coefficients in the sum are determined by equating the coefficients standing with the corresponding functions on both sides of the integral equation. The method presented in the paper allows treatment of the cases when the filament is inside the conductor and /or the currents have the same direction.**

***Index Terms*—current distribution; hollow conductor; filament; integral equation**

I. Introduction

Determination of current distribution in a system of parallel cylindrical conductors with time-varying currents is a very complex problem since this distribution in each of the conductors is not only affected by its own electromagnetic field (skin effect), but also by the fields of all other conductors (proximity effect). There are very few cases where this problem can be solved in a closed form, so generally an implementation of various numerical methods is required. Among the most commonly used approaches to the combined skin and proximity effects we mention here: usage of Maxwell's equations in terms of the magnetic vector potential [1–4], method of integral equations [5–11], boundary integral equation formulation [12,13], method of model functions [14], etc.

In this paper we use an integral equation to solve in a closed form the problem of low-frequency current distribution in a cylindrical hollow conductor in the presence of an outer current filament, the currents being assumed sinusoidal and to flow in opposite directions. The solution is sought in the form of an infinite sum of proper harmonics with some unknown coefficients. These coefficients are found in a closed form by equating coefficients with the corresponding functions on both sides of the integral equation.

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II. Integral Equation for the Current Density in a Hollow Cylindrical Conductor in the Presence of a Filament

Geometry of the problem is shown in Fig. 1. A current filament is parallel to a massive hollow cylindrical conductor of radii *a* and *b*.

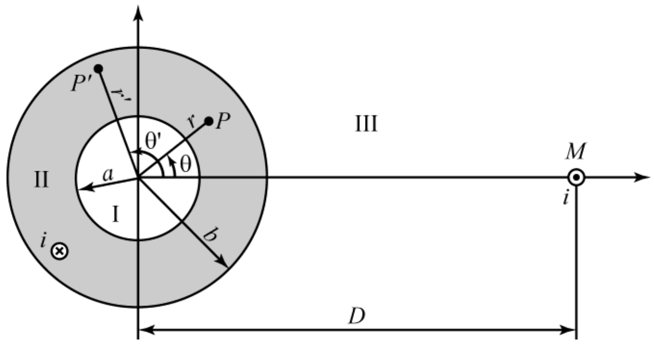

Fig. 1. Massive hollow cylindrical conductor and filament

The distance between the filament and the conductor axis is *D*, and the parameters of the conductor are σ and μ_0. Currents of equal r.m.s. *I* and of frequency *f* flow through the conductor and the filament in opposite directions. The objective is to find the current distribution in the massive conductor.

Following [5]-[6], [9], [11], we can write an integral equation for current density in the conductor:

$$J(r,\theta)=\frac{k^2}{4\pi}\left[\int_S r'\,J(r',\theta')\ln\frac{r^2+r'^2-2rr'\cos(\theta-\theta')}{D^2}\,dr'\,d\theta'-\right.$$
$$\left.-I\ln\frac{r^2+D^2-2rD\cos\theta}{D^2}\right]+K, \quad (1)$$

where $k^2=\mathrm{j}\omega\mu_0\sigma$, K is an unknown constant, and S is the annulus $a\le r\le b$, $0\le\theta\le 2\pi$.

Current density $J(r,\theta)$ in (1) is subject to the condition:

$$\int_S J(r,\theta)r\,dr\,d\theta = I. \quad (2)$$

III. Solution of the Integral Equation

Separation of variables in the wave equation for current density leads to the following particular solutions (harmonics): $I_n(kr)$, $K_n(kr)$, $\cos n\theta$, $\sin n\theta$, where I_n and K_n are

 ISBN 978-86-7466-930-3

modified Bessel functions. Symmetry requires that $J(r,\theta)$ be an even function in θ, which eliminates $\sin n\theta$. Hence, an appropriate form of the solution of (1) is

$$J(r,\theta)=\sum_{n=0}^{\infty}\left[A_n I_n(kr)+B_n K_n(kr)\right]\cos n\theta \tag{3}$$

with some unknown coefficients A_n and B_n. When (3) is substituted into (1), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\left[A_n I_n(kr)+B_n K_n(kr)\right]\cos n\theta=\\&=\frac{k^2}{4\pi}\sum_{n=0}^{\infty}A_n F_n(r,\theta)+\frac{k^2}{4\pi}\sum_{n=0}^{\infty}B_n G_n(r,\theta)+\\&+\frac{k^2 I}{2\pi}\sum_{n=1}^{\infty}\left(\frac{r}{D}\right)^n\frac{\cos n\theta}{n}+K\end{aligned} \tag{4}$$

where

$$F_n(r,\theta)=\int_S r' I_n(kr')\ln\frac{r^2+r'^2-2rr'\cos(\theta-\theta')}{D^2}dr'd\theta' \tag{5}$$

$$G_n(r,\theta)=\int_S r' K_n(kr')\ln\frac{r^2+r'^2-2rr'\cos(\theta-\theta')}{D^2}dr'd\theta' \tag{6}$$

and we made use of [15]

$$\ln\left(1+\frac{r^2}{D^2}-\frac{2r}{D}\cos\theta\right)=-2\sum_{n=1}^{\infty}\left(\frac{r}{D}\right)^n\frac{\cos n\theta}{n},\ \frac{r}{D}<1. \tag{7}$$

The double integrals F_n and G_n, given by (5) - (6), seemingly very complex, can be evaluated in a closed form. This is done in the Appendix for F_n; G_n is found in the same way. Here we state the result.

$$F_n(r,\theta)=\begin{cases}\dfrac{4\pi}{k}\left[b\ln\dfrac{b}{D}I_1(kb)-a\ln\dfrac{r}{D}I_1(ka)\right]-\dfrac{4\pi}{k^2}\left[I_0(kb)-I_0(kr)\right], n=0\\ \left\{-\dfrac{2\pi}{nk}\left[b\left(\dfrac{r}{b}\right)^n I_{n-1}(kb)-a\left(\dfrac{a}{r}\right)^n I_{n+1}(ka)\right]+\dfrac{4\pi}{k^2}I_n(kr)\right\}\cos n\theta,\ n>0\end{cases} \tag{8}$$

$$G_n(r,\theta)=\begin{cases}\dfrac{4\pi}{k}\left[a\ln\dfrac{r}{D}K_1(ka)-b\ln\dfrac{b}{D}K_1(kb)\right]-\dfrac{4\pi}{k^2}\left[K_0(kb)-K_0(kr)\right], n=0\\ \left\{-\dfrac{2\pi}{nk}\left[a\left(\dfrac{a}{r}\right)^n K_{n-1}(ka)-b\left(\dfrac{r}{b}\right)^n K_{n-1}(kb)\right]+\dfrac{4\pi}{k^2}K_n(kr)\right\}\cos n\theta,\ n>0\end{cases} \tag{9}$$

With F_n and G_n given by (8) – (9), we can equate the constant terms and the coefficients with $\ln(r/D)$, $r^n\cos n\theta$ and $r^{-n}\cos n\theta$ on both sides of (4), to get respectively

$$\begin{aligned}0=&A_0\left(kb\ln\frac{b}{D}I_1(kb)-I_0(kb)\right)+\\&+B_0\left(-kb\ln\frac{b}{D}K_1(kb)-K_0(kb)\right)+K\end{aligned} \tag{10}$$

$$0=-A_0 I_1(ka)+B_0 K_1(ka) \tag{11}$$

$$0=-A_n I_{n-1}(kb)+B_n K_{n-1}(kb)+\frac{kI}{\pi b}\left(\frac{b}{D}\right)^n \tag{12}$$

$$0=A_n I_{n+1}(ka)-B_n K_{n+1}(ka). \tag{13}$$

To determine the five unknown constants (A_0, B_0, A_n, B_n and K) we need one more equation beside the four equations given by (10) - (13). This additional equation is obtained from (2) if we replace $J(r,\theta)$ by (3). Then, we have

$$\begin{aligned}I&=\int_0^{2\pi}d\theta\int_a^b rJ(r,\theta)dr=\\&=\sum_{n=0}^{\infty}\int_0^{2\pi}\cos n\theta\int_a^b r\left(A_n I_n(kr)+B_n K_n(kr)\right)dr=\\&=A_0 2\pi\int_a^b rI_0(kr)dr+B_0 2\pi\int_a^b rK_0(kr)dr.\end{aligned} \tag{14}$$

The integrals in (14) are readily evaluated by making the change of variables $kr = x$ and using the identities [16]

$$xI_0(x)=(xI_1(x))'$$

$$xK_0(x)=-(xK_1(x))'.$$

Hence (14) becomes

$$I=\frac{2\pi}{k}\left\{A_0\left[bI_1(kb)-aI_1(ka)\right]+B_0\left[aK_1(ka)-bK_1(kb)\right]\right\}. \tag{15}$$

Now, (11) and (15) can be solved for A_0 and B_0

 ISBN 978-86-7466-930-3

$$A_0 = \frac{kI}{2\pi b}\frac{K_1(ka)}{I_1(kb)K_1(ka)-I_1(ka)K_1(kb)} \tag{16}$$

$$B_0 = \frac{kI}{2\pi b}\frac{I_1(ka)}{I_1(kb)K_1(ka)-I_1(ka)K_1(kb)} \tag{17}$$

and A_n, B_n are found from (12) - (13)

$$A_n = \frac{kI}{\pi b}\left(\frac{b}{D}\right)^n \frac{K_{n+1}(ka)}{K_{n+1}(ka)I_{n-1}(kb)-K_{n-1}(kb)I_{n+1}(ka)} \tag{18}$$

$$B_n = \frac{kI}{\pi b}\left(\frac{b}{D}\right)^n \frac{I_{n+1}(ka)}{K_{n+1}(ka)I_{n-1}(kb)-K_{n-1}(kb)I_{n+1}(ka)}\,. \tag{19}$$

The remaining unknown K can be determined from (10), (16) – (17), but it is of no importance.

Finally, (3) and (16) - (19) determine current density in the massive conductor

$$\begin{aligned} J(r,\theta) &= \frac{kI}{2\pi b}\frac{K_1(ka)I_0(kr)+I_1(ka)K_0(kr)}{I_1(kb)K_1(ka)-I_1(ka)K_1(kb)} + \\ &+ \frac{kI}{\pi b}\sum_{n=1}^{\infty}\left(\frac{b}{D}\right)^n \frac{K_{n+1}(ka)I_n(kr)+I_{n+1}(ka)K_n(kr)}{I_{n-1}(kb)K_{n+1}(ka)-I_{n+1}(ka)K_{n-1}(kb)}\cos n\theta. \end{aligned} \tag{20}$$

Physically, the first term on the right-hand side of (20) is due to the skin effect, and the infinite sum accounts for the influence of the filament (proximity effect).

The same method applies if the filament is inside the conductor. In this case b/D in the infinite sum in (20) should be replaced by D/b.

Expression (20) may also be derived by using Maxwell's equations [4]. In this approach the magnetic vector potential is determined in media I, II and III (Fig. 1), which includes usage of appropriate boundary conditions on the interfaces $r=a$ and $r=b$. The approach in the present paper is much simpler –solving integral equation (1) does not involve any boundary conditions.

It may be shown that the particular cases $a = 0$ (massive conductor [6], [11]), and $b - a = d << a$ (thin tubular conductor [5], [9]), follow from (20). Derivation for the former case is straightforward; for the latter case, it requires usage of a few formulas that involve Bessel functions.

IV. CONCLUSION

In this paper, we derived a closed-form solution for the current distribution in a hollow cylindrical conductor in the presence of a current filament placed outside the conductor. A solution of an integral equation for the current density is found in the form of an infinite sum of proper harmonics - the modified Bessel functions of the second kind and the trigonometric functions. A remarkable feature of the method used is that it does not involve any boundary conditions.

APPENDIX

In this Appendix we prove (8), relation (9) is justified by the same procedure.

Let $n = 0$. Then

$$\begin{aligned} F_0(r,\theta) &= \int_0^{2\pi} a\theta' \int_a^b r' I_0(kr') \ln\frac{r^2+r'^2-2rr'\cos(\theta-\theta')}{D^2}dr' = \\ &= \int_0^{2\pi} a\theta' \left[\int_a^r r' I_0(kr') \ln r^2 \frac{r^2+r'^2-2rr'\cos(\theta-\theta')}{r^2 D^2}dr' + \int_r^b r' I_0(kr') \ln r'^2 \frac{r^2+r'^2-2rr'\cos(\theta-\theta')}{r'^2 D^2}dr'\right] = \\ &= \int_0^{2\pi} a\theta' \left\{\int_a^r r' I_0(kr')\left[\ln\frac{r^2}{D^2}+\ln\left(1+\frac{r'^2}{r^2}-2\frac{r'}{r}\cos(\theta-\theta')\right)dr'\right] + \int_r^b r' I_0(kr')\left[\ln\frac{r'^2}{D^2}+\ln\left(1+\frac{r^2}{r'^2}-2\frac{r'}{r}\cos(\theta-\theta')\right)dr'\right]\right\} \\ &= \int_0^{2\pi} a\theta' \left[\ln\frac{r^2}{D^2}\int_a^r r' I_0(kr')dr - 2\int_a^r r' I_0(kr')\sum_{m=1}^{\infty}\left(\frac{r'}{r}\right)^m \frac{\cos m(\theta-\theta')}{m}dr' + \int_r^b r' \ln\frac{r'^2}{D^2} I_0(kr')dr' - 2\int_r^b r' I_0(kr')\sum_{m=1}^{\infty}\left(\frac{r}{r'}\right)^m \frac{\cos m(\theta-\theta')}{m}dr'\right] \end{aligned} \tag{A1}$$

where we used (7). We do not need to evaluate the integrals that include infinite sum, since $\cos m(\theta-\theta') = \cos m\theta \cos m\theta' + \sin m\theta \sin m\theta'$, and the subsequent integration in θ' gives zero, since $\int_0^{2\pi}\cos m\theta' d\theta' = \int_0^{2\pi}\sin m\theta' d\theta' = 0,\ m \geq 1$. The two remaining integrals are evaluated as follows

$$\int_a^r r' I_0(kr')dr' = \frac{1}{k^2}\int_{ka}^{kr} u I_0(u)du = \frac{1}{k}\left(rI_1(kr) - aI_1(ka)\right), \tag{A2}$$

where the change of variables $kr'=u$ and the identity

$$uI_0(u) = (uI_1(u))',$$

were used. Similarly, by using integration by parts

 ISBN 978-86-7466-930-3

$$\begin{aligned}&\int_r^b r'\ln\frac{r'^2}{D^2}I_0(kr')dr'=\frac{2}{k^2}\int_{kr}^{kb}u\ln\frac{u}{kD}I_0(u)du=\\&=\frac{2}{k^2}\left[uI_1(u)\ln\frac{u}{kD}\Big|_{kr}^{kb}-\int_{kr}^{kb}uI_1(u)\frac{du}{u}\right]=\\&=\frac{2}{k^2}\left(kb\ln\frac{b}{D}I_1(kb)-kr\ln\frac{r}{D}I_1(kr)-\int_{kr}^{kb}I_1(u)du\right)=\\&=\frac{2}{k^2}\left[kb\ln\frac{b}{D}I_1(kb)-kr\ln\frac{r}{D}I_1(kr)-\left(I_0(kb)-I_0(kr)\right)\right]\end{aligned}\tag{A3}$$

since [16]

$$I_1(u)=I_0'(u).$$

Now, relation (8) for $n = 0$ follows from (A1) - (A3).

For $n \geq 1$

$$\begin{aligned}F_n(r,\theta)=\int_0^{2\pi}\cos n\theta' d\theta'\Bigg[&\ln\frac{r^2}{D^2}\int_a^r r'I_n(kr')dr'-\\&-2\int_a^r r'I_n(kr')\sum_{m=1}^{\infty}\left(\frac{r'}{r}\right)^m\frac{\cos m(\theta-\theta')}{m}dr'+\\&+\int_r^b r'ln\frac{r'^2}{D^2}I_n(kr')dr'-2\int_r^b r'I_n(kr')\sum_{m=1}^{\infty}\left(\frac{r}{r'}\right)^m\frac{\cos m(\theta-\theta')}{m}dr'\Bigg]d\theta'\end{aligned}\tag{A4}$$

In this case, the integrals $\int_a^r r'I_n(kr')dr'$ and $\int_r^b r'\ln\frac{r'^2}{D^2}I_n(kr')dr'$ are irrelevant, since the subsequent integration in θ' gives zero $\left(\int_0^{2\pi}\cos n\theta'd\theta'=0,\ n\geq 1\right)$, hence (A4) becomes

$$\begin{aligned}F_n(r,\theta)=&-2\sum_{m=1}^{\infty}\frac{1}{mr^m}\int_0^{2\pi}\cos n\theta'\cos m(\theta-\theta')d\theta'\int_a^r r'^{m+1}I_m(kr')dr'-\\&-2\sum_{m=1}^{\infty}\frac{r^m}{m}\int_0^{2\pi}\cos n\theta'\cos m(\theta-\theta')d\theta'\int_r^b r'^{-m+1}I_m(kr')dr'.\end{aligned}\tag{A5}$$

Only the term with $m = n$ should be kept in the infinite sums, since for $m \neq n$

$$\int_0^{2\pi}\cos n\theta'\cos m(\theta-\theta')d\theta'=$$

$$=\cos n\theta\int_0^{2\pi}\cos n\theta'\cos m\theta'd\theta'+\sin m\theta\int_0^{2\pi}\cos n\theta'\sin m\theta'd\theta'=0,$$

due to orthogonality of the sine and cosine functions. As a consequence, (A5) simplifies to

$$\begin{aligned}F_n(r,\theta)=\Bigg(&-\frac{2\pi}{nr^n}\int_a^r r'^{n+1}I_n(kr')dr'-\\&-\frac{2\pi r^n}{n}\int_r^b r'^{-n+1}I_n(kr')dr'\Bigg)\cos n\theta\end{aligned}\tag{A6}$$

where we have taken into account that

$$\int_0^{2\pi}\cos n\theta'\cos n(\theta-\theta')d\theta'=$$

$$=\cos n\theta\int_0^{2\pi}\cos^2 n\theta'd\theta'+\sin n\theta\int_0^{2\pi}\cos n\theta'\sin n\theta'd\theta'=\pi\cos n\theta.$$

The change of variables $kr'=u$ and the relations [16]

$$u^{n+1}I_n(u)=\left(u^{n+1}I_{n+1}(u)\right)'$$
$$u^{-n+1}I_n(u)=\left(u^{-n+1}I_{n-1}(u)\right)'$$

enable to evaluate the two integrals in (A6)

$$\int_a^r r'^{n+1}I_n(kr')dr'=\frac{1}{k}\left(r^{n+1}I_{n+1}(kr)-a^{n+1}I_{n+1}(ka)\right)\tag{A7}$$

$$\int_r^b r'^{-n+1}I_n(kr')dr'=\frac{1}{k}\left(b^{-n+1}I_{n-1}(kb)-r^{-n+1}I_{n-1}(kr)\right).\tag{A8}$$

Finally, from (A6) - (A8)

$$\begin{aligned}F_n(r,\theta)=&-\frac{2\pi}{nk}\left\{r[I_{n+1}(kr)-I_{n-1}(kr)]+b\left(\frac{r}{b}\right)^n I_{n+1}(kb)-\right.\\&\left.-a\left(\frac{a}{r}\right)^n I_{n+1}(ka)\right\}\cos n\theta=\\&=\left\{-\frac{2\pi}{nk}\left[b\left(\frac{r}{b}\right)^n I_{n-1}(kb)-a\left(\frac{a}{r}\right)^n I_{n+1}(ka)\right]+\frac{4\pi}{k^2}I_n(kr)\right\}\cos n\theta\end{aligned}$$

since [16]

$$I_{n+1}(kr)-I_{n-1}(kr)=-\frac{2n}{kr}I_n(kr).$$

Therefore, the proof of (8) is completed.

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