Allpass Based Double Notch IIR Filters with Constant Phase

Goran Stančić, Ivana Kostić and Stevica Cvetković

***Abstract*—Narrow stopband filters with two notch frequencies and piecewise constant phase are investigated in this paper. The notch filters are determined by allpass subfilter phase approximation. Obtained filters with simple and double poles are compared in conditions when fractional part of the coefficients is represented with limited number of bits.**

***Index Terms*—Notch IIR filters, allpass filters, phase approximation, constant phase, quantization.**

I. Introduction

THE ideal notch filters have exactly zero magnitude at frequency which need to be removed from input signal spectrum and unity gain otherwise. In many devices in practice the power line frequency signal and corresponding harmonics are often treated as noise [1]. Over time, digital electronic components become faster and cheaper allowing designers to oversample input signal. In that case the neighboring harmonics start to go closer to one another at the frequency axis. Notch filters are part of radar systems, control and instrumentation systems, medical applications and communications systems. In order to keep the distortion of desired signal as low as possible, the stopband of the notch filter should be as narrow as possible. In this paper problems associated with close notch frequencies will be observed. All of the presented results are given for double notch filters but it is easy to modify proposed method for arbitrary number of notch frequencies.

II. Realization structure

In addition to standard realization structures filters can be obtained by parallel connection of two allpass filters [2]. The magnitude of resulting filter depends on phases of applied allpass filters. That is a reason why design of the linear/constant phase IIR filters comes down to the allpass phase approximation problem.

In practice linear and constant phase are ultimate goals to avoid phase distortions [3]. To obtain the notch filter with approximately constant phase one allpass filter becomes direct path as shown in Fig. 1 [4]. The notch filter with linear phase will be achieved if one allpass filter is pure delay [5].

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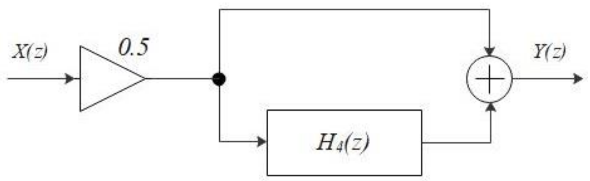

Fig. 1. Double notch filter realised as parallel connection of direct path and fourth order allpass filter.

The transfer function of constant phase notch filter is

$$H(z) = 0.5(1 + H_4(z)) \tag{1}$$

where $H_4(z)$ represents transfer function of allpass filter of the form

$$H_4(z) = \prod_{i=1}^{2} \frac{(\rho_i - e^{-j\theta} z^{-1})(\rho_i - e^{j\theta} z^{-1})}{(1 - \rho_i e^{j\theta} z^{-1})(1 - \rho_i e^{-j\theta} z^{-1})} \tag{2}$$

taking into account the fact that allpass filters have conjugate-reciprocal pole-zero pairs. Magnitude of the notch filter directly depends on the allpass filters phase φ

$$\left|H(e^{j\omega})\right| = \left|cos\frac{\varphi(\boldsymbol{\rho}, \boldsymbol{\theta}, \omega)}{2}\right| \tag{3}$$

where $\boldsymbol{\rho}$ and $\boldsymbol{\theta}$ represent moduli and phase angles of the allpass filters poles, respectively. Every pole and zero contribute to the phase with $\pi/2$ radians making fourth order filter to reach -4π radians phase at Nyquist frequency as shown in Fig. 2.

The closer a pole is to the unit circle the higher negative slope is at frequencies in vicinity of pole position. The phase is monotonically decreasing function of frequency with emphasized jump around pole position. Fourth order transfer function also could be obtained with two simple poles. This case is marked with d) in Fig. 2. Now poles are not at the same frequency and two separate phase jumps of approximately -2π radians could be observed.

According to (3), filter realized with described parallel structure possess passbands at frequencies ω where $\varphi(\boldsymbol{\rho}, \boldsymbol{\theta}, \omega)$ approximates $2k\pi$ with allowed tolerance ε, for $k \in \mathbb{Z}$. Stopbands would be obtained at frequencies where phase value is approximately $2(k+1)\pi$.

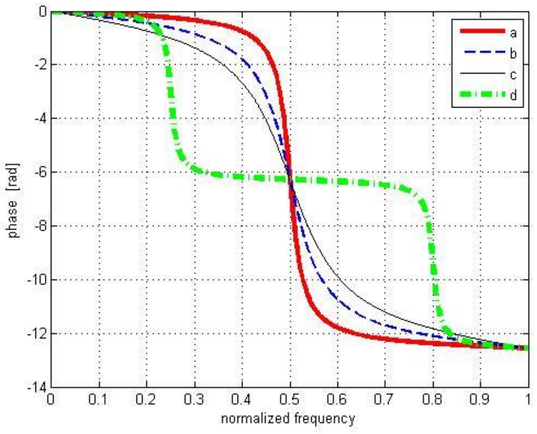

Fig. 2. Phase of allpass filter of fourth order for double pole moduli a) ρ=0.94, b) ρ=0.85, c) ρ=0.77 for θ=0.5π and d) simple poles $\rho_1 = 0.968, \rho_2 = 0.967$ for $\theta_1 = 0.25\pi$ and $\theta_2 = 0.8\pi$.

For predefined attenuation a in decibels, at stopbands or passbands boundary frequencies, allowed phase approximation error has value

$$\varepsilon = 2\arccos\left(10^{-a/20}\right) \tag{4}$$

III. Design procedure

If filter with two notch frequencies has double pole, to determine the transfer function only two unknown values need to be calculated- modulus and angle of the pole. The positions of notch frequencies are of highest importance, so one need to solve next system of equations

$$\begin{aligned}\varphi(\rho,\theta,\omega_{n1}) &= -\pi\\ \varphi(\rho,\theta,\omega_{n2}) &= -3\pi.\end{aligned} \tag{5}$$

Notch filter with two simple poles has four unknown parameters. It allows two boundary frequencies to be controled. System of equations that provide notch filter transfer function has now form

$$\begin{aligned}\varphi(\boldsymbol{\rho},\boldsymbol{\theta},\omega_{n1}) &= -\pi\\ \varphi(\boldsymbol{\rho},\boldsymbol{\theta},\omega_{n2}) &= -3\pi\\ \varphi(\boldsymbol{\rho},\boldsymbol{\theta},\omega_{l1}) &= -\varepsilon\\ \varphi(\boldsymbol{\rho},\boldsymbol{\theta},\omega_{r3}) &= -4\pi+\varepsilon\end{aligned} \tag{6}$$

where ω_{l1} and ω_{r3} represents boundary frequencies of the first and the third passband, respectively.

Instead of passband edges, it is possible to use stopband edges in system (6) but in (4) minimal attenuation in stopband need to be applied, with minimal modifications of last two equations in (6) ($-\varepsilon$ will be changed with $-\pi \pm \varepsilon$ and $-4\pi + \varepsilon$ with $-3\pi \pm \varepsilon$).

Systems of equations (5) and (6) could be solved applying some iterative procedure which demands the initial solution. System (6) is given in alternative form

$$\boldsymbol{A\Delta} = \boldsymbol{B} \tag{7}$$

after approximating phase $\varphi(\boldsymbol{\rho},\boldsymbol{\theta},\omega)$ by truncated Taylor series, where elements of matrix $\boldsymbol{A}$ are

$$a_{ij} = \begin{cases}\dfrac{d\varphi(\omega_i)}{d\rho_j}, & j = 1{,}2 \quad i = 1,..,4\\ \dfrac{d\varphi(\omega_i)}{d\theta_{j-2}}, & j = 3{,}4 \quad i = 1,..,4\end{cases} \tag{8}$$

elements of column vector $\boldsymbol{B}$ are

$$\boldsymbol{B} = \begin{bmatrix}-\pi - \varphi(\boldsymbol{\rho}^*,\boldsymbol{\theta}^*,\omega_{n1})\\ -3\pi - \varphi(\boldsymbol{\rho}^*,\boldsymbol{\theta}^*,\omega_{n2})\\ -\varepsilon - \varphi(\boldsymbol{\rho}^*,\boldsymbol{\theta}^*,\omega_{l1})\\ -4\pi + \varepsilon - \varphi(\boldsymbol{\rho}^*,\boldsymbol{\theta}^*,\omega_{r3})\end{bmatrix} \tag{9}$$

and vector of increments to be found is given with

$$\boldsymbol{\Delta} = [\Delta\rho_1;\ \Delta\rho_2;\ \Delta\theta_1;\ \Delta\theta_2] \tag{10}$$

In every iterative step system (7) is solved, the modulus and phase angle of poles are corrected until maximal absolute value of elements of column vector $\boldsymbol{\Delta}$ becomes less than predefined small value (in all given examples 10^{-10} is applied). As a good initial solution one could choose values

$$\rho^* = 0.9 \quad \text{and} \quad \theta^* = \frac{\omega_{n1} + \omega_{n2}}{2} \tag{11}$$

for double pole and

$$\boldsymbol{\rho}^* = [0.9; 0.9] \quad \text{and} \quad \boldsymbol{\theta}^* = [\omega_{n1};\ \omega_{n2}] \tag{12}$$

for simple poles. Extensive experiments shown that the final solution would be reached in less than ten iterations for arbitrary feasible input parameters.

IV. RESULTS

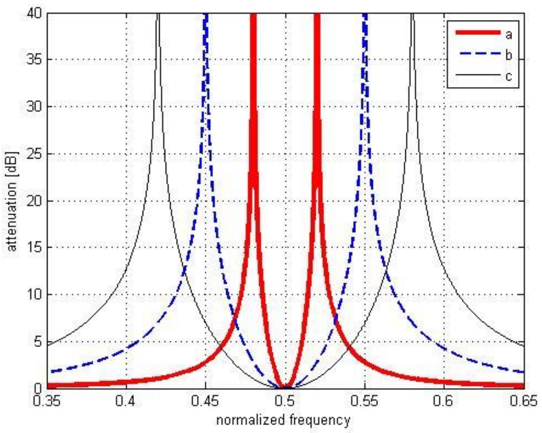

Fig. 3. The filters with two notch frequencies realized with double pole for a) $\omega_{n1} = 0.48\pi, \omega_{n2} = 0.52\pi$, b) $\omega_{n1} = 0.45\pi, \omega_{n2} = 0.55\pi$ and c) $\omega_{n1} = 0.42\pi, \omega_{n2} = 0.58\pi$.

In Fig. 3 are displayed characteristics of double notch filter attenuation for different zero magnitude frequencies. All filters have a double pole. Taking into account the phase characteristics shown in Fig. 2, to achieve more distance between the notches it is inevitable to move the pole closer to the origin. That is good for the stability because the pole moves further from the unit circle. Lower pole modulus values provoke lower phase slope, so the transition zones and stopbands become wider at the expense of passbands. This feature points to fact that double pole notch filter has restricted application. It is not possible to choose higher order allpass filter in attempt to improve notch filters characteristics, as in case of linear phase filters.

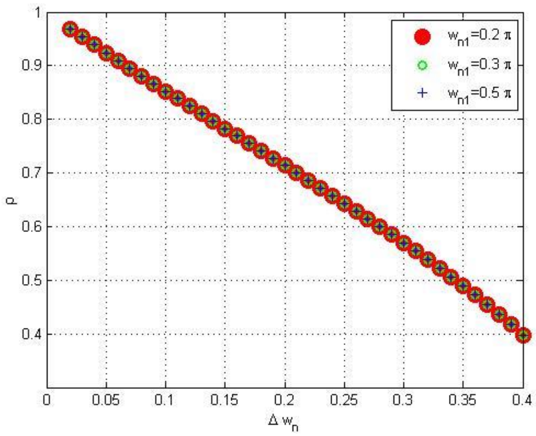

Fig. 4. The dependance of double pole modulus on notch frequencies gap for different ω_{n1} locations ($\Delta\omega_n = \omega_{n2} - \omega_{n1}$).

From Fig. 4 could be observed that most significant influence on double pole modulus has the notch frequencies gap. The very value of notch frequencies location have no visible impact. In practical realization of digital filter, the number of bits for filters coefficients representation need to be defined.

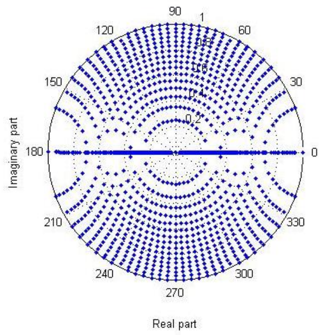

Fig. 5. Possible positions of filters poles for fixed point arithmetics when 4 bits are reserved for fractional part of transfer function coefficients.

Finite number of bits leads to rounded values of coefficients so realized filter characteristics just approximate derived ones. Possible positions of poles of the second order transfer function are displayed in Fig. 5 in case four bits are dedicated to fractional parts. In other words, calculated transfer function will be replaced with approximated one and obtained poles have to move from obtained positions to available locations like in Fig. 5.

The Fig. 5 indicates the fact that one can expect bigger error as consequence of quantization if notches are positioned at low and high frequencies.

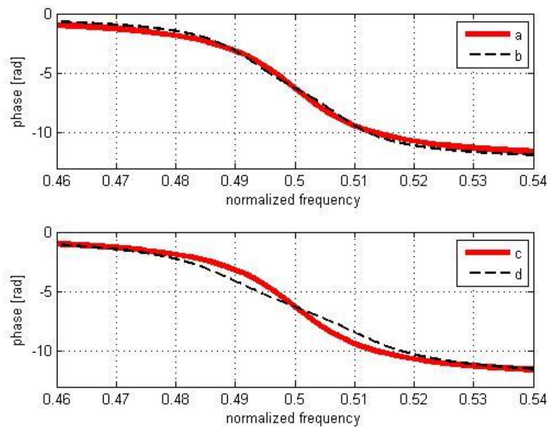

Fig. 6. Phase of notch filters ($\omega_{n1} = 0.49\pi$, $\omega_{n2} = 0.51\pi$) with a),c) double and b), d) simple poles before and after quantization fractional parts with 4 bits, respectively.

As first example filters with $\omega_{n1} = 0.49\pi$ and $\omega_{n2} = 0.51\pi$ are designed with desribed procedure. Attenuation of 1 dB is chossen in passbands. Boundary frequencies are ω_{l1}=0.47 π and ω_{r3}=0.53 π. Corresponding phase, attenuation and poles location are presented in Fig. 6, Fig. 7 and Fig. 8, respectively.

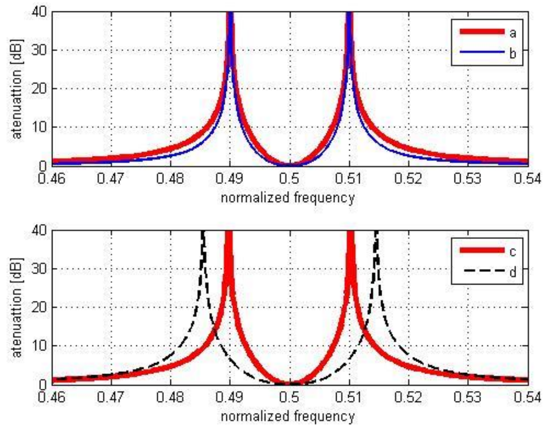

Fig. 7. Attenuation of notch filters ($\omega_{n1} = 0.49\pi$, $\omega_{n2} = 0.51\pi$) with a), c) double and b), d) simple poles before and after quantization fractional parts with 4 bits, respectively.

As it was expected, the phase undergo changes as repercussion of quantization, causing notch frequencies to displace. The notches are misplaced for 0.0142 for simple poles and $9 \cdot 10^{-4}$ for double pole case. Symmetry helps double pole filter less to degrade. The reason can be found in Fig. 5, where one can observe that pole with θ=0.5π will change only pole modulus as given in Fig. 8. Simple poles changed both moduli and phase angles causing significant mismatch between desired and obtained notches.

All obtained filters are realized as serially-cascaded second-order sections. Denominator coefficients of second order sections of the allpass filter with simple poles are given in Table I. All presented results are obtained in Matlab. The coefficients of allpass filter with double pole are presented in Table II.

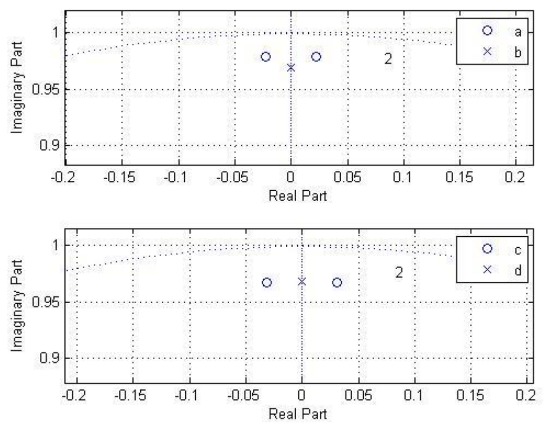

Fig. 8. Location of notch filters poles ($\omega_{n1} = 0.49\pi$, $\omega_{n2} = 0.51\pi$) with a), c) simple and b), d) double poles before and after quantization fractional parts with 4 bits, respectively.

TABLE I
COEFFICIENTS OF THE SECOND ORDER SECTIONS (SIMPLE POLES)

1.	0.0451	0.9581
1.	-0.0451	0.9581

TABLE II
COEFFICIENTS OF THE SECOND ORDER SECTIONS (DOUBLE POLE)

1.	0.	0.9391
1.	0.	0.9391

After quantization, with four bits dedicated to the fractional part, new values for second order sections coefficients are obtained as given in Table III. Table IV contains coefficients of allpass filter with a double pole. Because of existing symmetry second order sections have one coefficient equal to zero demanding less multipliers and adders in hardware realization.

TABLE III
COEFFICIENTS OF THE SECOND ORDER SECTIONS AFTER QUANTIZATION (SIMPLE POLES)

1.	0.0625	0.9375
1.	-0.0625	0.9375

TABLE IV
COEFFICIENTS OF THE SECOND ORDER SECTIONS AFTER QUANTIZATION (DOUBLE POLE)

1.	0.	0.9375
1.	0.	0.9375

For second example filters with $\omega_{n1} = 0.10\pi$ and $\omega_{n2} = 0.11\pi$ are chosen. For design procedure values ω_{l1}=0.09 π, ω_{r3}=0.12 π and a =3 dB are adopted. Obtained phase, attenuation and poles location are presented in Fig. 9, Fig. 10 and Fig. 11, respectively. These filters have poles in area where possible pole locations are scattered. As consequence, quantization of filter coefficients will seriously degrade characteristics. Close notches demand poles to be near the unit circle to provide enough steep slope.

Even 5 bits dedicated to the fractional part was not enough to stop a pole at the end of the unit circle. From (2) it is obvious that in such a case influence of the allpass zero and pole is identical, forcing the transfer function to degrade to second order. As a result, filter possess only one notch, as given in Fig. 10 d). The double pole filter has two notches after quantization but rear possible pole positions considerably influence the notches to displace.

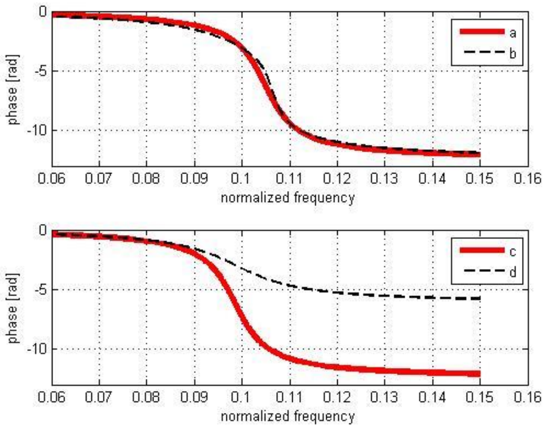

Fig. 9. Phase of notch filters ($\omega_{n1} = 0.10\pi$, $\omega_{n2} = 0.11\pi$) with a),c) double and b), d) simple poles before and after quantization fractional parts with 5 bits,respectively.

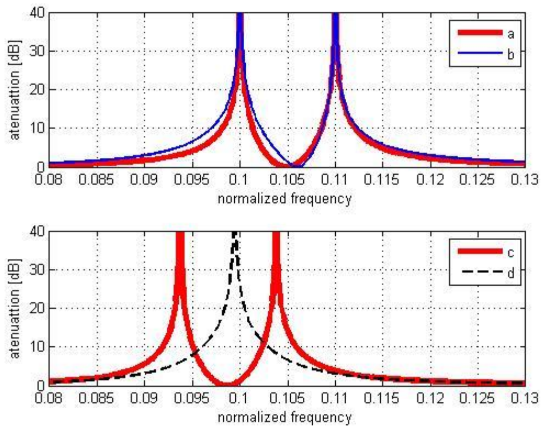

Fig. 10. Attenuation of notch filters ($\omega_{n1} = 0.10\pi$, $\omega_{n2} = 0.11\pi$) with a), c) double and b), d) simple poles before and after quantization of fractional parts with 5 bits, respectively.

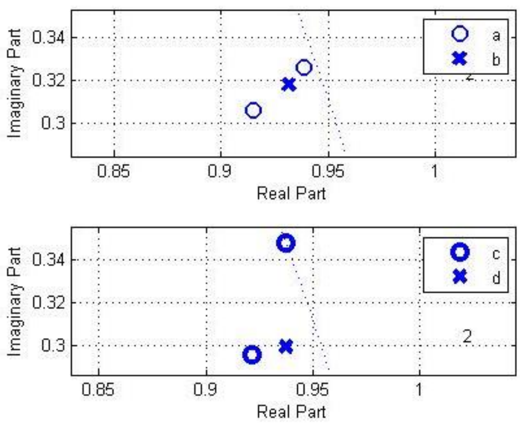

Fig. 11. Location of notch filters poles ($\omega_{n1} = 0.10\pi$, $\omega_{n2} = 0.11\pi$) with a), c) simple and b), d) double poles before and after quantization fractional parts with 5 bits, respectively.

V. Conclusion

Double notch filters with constant phase in passbands are investigated in this paper. Two similar solutions are compared and impact of quantization is analyzed. Parallel allpass structure guarantee low passband sensitivity. Quantization effects primarily affect notch filters stopband, moving away locations of notches from desired positions. The double pole filters are not good choice in case when gap between notches is wider than 0.2π because low phase slope causes transition zones to spread, degrading selectivity. On the other hand, the pole has lower modulus if filter possess double pole, what is guarantee to remain stable after quantization and still to have both notches. The distance of simple pole from the unit circle is always smaller compared to double pole. As consequence, after quantization if notches are close to each other it may occur one or both simple poles to finish at the unit circle and quantized version of filter lose selectivity. Design of numerous notch filters with different notches location have been shown that double pole solution is better option for close notches and small number of bits dedicated to the coefficients fractional part.

Acknowledgment

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