Modelling and Control of a Series Direct Current (DC) Machines Using Feedback Linearization Approach

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Abstract—The nonlinear feedback control system applied to the direct current - DC motor is proposed in this research. Nonlinear mathematical model has been obtained using dead zone, Coulomb and viscous friction. The system stability has been analyzed using Lyapunov stability theory. The effectiveness and the comparison of the performance between linear and nonlinear control algorithm have been validated using Matlab/Simulink software. From the conclusions, based on the simulation and experimental results that have been provided, it is easy to see that nonlinear control systems are more suitable and have a better reach for controlling position. The validity of using feedback linearization in DC motors has been proven.

Index Terms—feedback linearization; nonlinear systems; nonlinear control; identification

I. INTRODUCTION

Position control in a direct current motor (DC) has been one of the most fundamental and challenging tasks, that has been largely studied for decades. Many studies have been done to model electrical machines. For example, serial DC motor has often been modeled as linear object. On the other hand, models in which motor current or flux are found as essential parameters are considered to be nonlinear [1]. This paper presents the design and implementation concerning both, linear and nonlinear models for the system. They are obtained for identification and control purposes. The major nonlinearities in the system, such as Coulomb friction and dead zone, are investigated and integrated in the nonlinear model [2]. In the different types of application accurate control of position in DC machines is a great challenge for engineers. Disparate controllers have been proposed to lead the position of DC machines into the desired value. For example Proportional-Integral-Derivative (PID) controller is a popular controller in industries due to simple structure, low cost and easy to implement. It provides reliable performance for the system if PID parameter is identified properly. But it suffers due to lack of robustness [1]. The linear approximation, of the nonlinear state space representation of the series DC motor, around the equilibrium point and PI controller design the tracking performance is deteriorated in the periods in which the speed is reduced. This is due to the fact that the input signal $u(t)$ is limited to a minimum of 0 [V]. That is, in this condition the motor is actually operating in open loop [3].

Besides linear, there are plenty of nonlinear controllers: the fuzzy logic and genetic – based new fuzzy models [4], artificial neural networks [5], adaptive control technique [6], and others.

It is important to make this comparison to find out under what conditions a technique presents a superior performance over the other one and thus have the certainty when it is useful to implement nonlinear controllers, which have greater complexity [7].

Modelling a nonlinearity is often a very complicated challenge. One of the first steps in the synthesis of a control system is to create a mathematical model, because it saves time and it brings the cost-effectiveness.

The main objective of this research is the development and later implementation of a nonlinear control system, by the feedback linearization method, for a laboratory installed DC motor, SRV02 Rotary Servo Base Unit, which has been considered as a single-input-single-output (SISO) system.

Feedback linearization is an approach to nonlinear control design which has attracted a great deal of research interest in recent years. By a combination of a nonlinear transformation and state feedback (feedback linearization), the nonlinear control design is reduced to designing a linear control law [8]. The central idea of the approach is to algebraically transform a nonlinear system dynamics into a (fully or partly) linear one, so that linear control techniques can be applied. This differs entirely from conventional linearization in that feedback linearization is achieved by exact state transformations and feedback, rather than by linear approximations of the dynamics [9]. This technique has been successfully implemented in many applications of control, such as industrial robots, high performance aircraft, helicopters and biomedical dispositifs, more tasks used the methodology are being now well advanced in industry [10].

II. LINEAR MODEL OF SYSTEM DYNAMICS

Constructing an accurate model is a pivotal stage in practical control problems. An appropriately developed system model is essential for reliability of the designed control. A DC series motor is an example of a simple, controlled process that can serve as a vehicle for the evaluation of the performance of the various controllers [4].

A schematic diagram of the DC motor is given in Fig. 1.
The equations that describe the motor electrical components are as follows:

\[ V_m(t) = R_m I_m(t) + L_m \frac{dI_m(t)}{dt} + e_b(t) \]  \hspace{1cm} (1)

\[ e_b(t) = k_m \omega_m(t) \]  \hspace{1cm} (2)

where \( V_m, e_b, k_m \) and \( \omega_m \) are motor voltage, back electromotive voltage, back electromotive voltage constant and speed of the motor shaft, respectively. Since the motor inductance \( L_m \) is much less than its resistance \( R_m \), it can be ignored [11]. Solving the system of equations for motor current \( I_m \), we get an electrical equation of DC motor:

\[ I_m(t) = \frac{V_m(t) - k_m \omega_m(t)}{R_m} \]  \hspace{1cm} (3)

The linear model can be obtained using the Second Newton’s Law of Motion and connection between moment of inertia of the load \( J_l \) and of the motor shaft \( J_m \), speed of the load shaft \( \omega_l \), viscous friction acting on the motor shaft \( B_m \) and on the load shaft \( B_l \), total torque applied on the load \( \tau_l \) and on the motor \( \tau_m \), with resulting torque acting on the motor shaft from the load torque denoted as \( \tau_{ml} \):

\[ J_l \frac{d\omega_l(t)}{dt} + B_l \omega_l(t) = \tau_l(t) \]  \hspace{1cm} (4)

\[ J_m \frac{d\omega_m(t)}{dt} + B_m \omega_m(t) + \tau_m(t) = \tau_{ml}(t) \]  \hspace{1cm} (5)

so the mechanical equation is:

\[ J_{eq} \frac{d\omega_l(t)}{dt} + B_{eq} \omega_l(t) = \eta_g K_g \tau_m(t) \]  \hspace{1cm} (6)

where \( J_{eq} \) and \( B_{eq} \) are total moment of inertia and damping term. \( \eta_g \) and \( K_g \) are, respectively, the gearbox efficiency and the total gear ratio.

Combining electrical and mechanical equations, assuming that motor torque is proportional to the voltage, the final equation becomes:

\[ \left( \frac{d}{dt} \omega_l(t) \right) J_{eq} + B_{eq} \omega_l(t) = A_m V_m(t) \]  \hspace{1cm} (7)

where the equivalent damping term is given by:

\[ B_{eq,v} = \frac{\eta_g K_g^2 \eta_m k_m + B_{eq} R_m}{R_m} \]  \hspace{1cm} (8)

where the \( \eta_m \) and \( k_m \) are the motor efficiency and the current-torque constant respectively. The actuator gain equals:

\[ A_m = \frac{\eta_g K_g \eta_m k_m}{R_m} \]  \hspace{1cm} (9)

Linear mathematical model that defines the relationship between voltage and angular position of the load shaft \( \theta_l \) is:

\[ J_{eq} \ddot{\theta}_l(t) + B_{eq,v} \dot{\theta}_l(t) = A_m V_m(t). \]  \hspace{1cm} (10)

Choosing \( y = \theta_l \) as output variable and \( u = V_m \) as input signal, state equation of the system is obtained as follows:

\[ J_{eq} \ddot{y}(t) + B_{eq,v} \dot{y}(t) = A_m u(t). \]  \hspace{1cm} (11)

III. EXPERIMENTAL VERIFICATION OF THE OBTAINED LINEAR MATHEMATICAL MODEL

Responses of the system represented with the block diagram in the Fig. 2 are shown in the Fig. 3 and Fig. 4. After recording the responses of the object, comparisons were made with the responses obtained by simulations of the linear model, for step and sinusoidal inputs.

![块图图示](https://example.com/figure2.png)

**Fig. 2.** Block diagram of a linear system

![实验结果图示](https://example.com/figure3.png)

**Fig. 3.** Experimental results: comparison between real and model data for step input
AUI 2.2.3

Fig. 4. Experimental results: comparison between real and model data for sinusoidal input

From this simulated example, an important conclusion can be drawn. Simulated linear model of the plant does not match well response of the real system. It is obvious that mathematical model of the series DC motor is nonlinear.

IV. FEEDBACK LINEARIZATION

In this section, the conditions for the linearizing transformation and nonlinear feedback allowing the DC motor to be controlled are outlined. Of particular interest will be the coordinate transformation also known as diffeomorphism, and the feedback law which will allow it to be accomplished.

Feedback linearization approach differs from the classical linearization (about the desired equilibrium point) in that no approximation is used; it is exact. Exactness, however, assumes perfect knowledge of the state equation and uses that knowledge to cancel the nonlinearities of the system. Since perfect knowledge of the state equation and exact mathematical cancellation of terms are almost impossible, the implementation of this approach will almost always result in a close-loop system, which is a perturbation of a nominal system whose origin is exponential stable. The validity of the method draws upon Lyapunov theory for perturbed systems whose origin is exponential stable. It is obvious that implementation of this approach will almost always result in a feedback law which will allow it to be accomplished.

Consider the single – input – single – output nonlinear SISO system [12]:

\[ \dot{x} = f(x) + g(x)u \quad y = h(x) \]  \hspace{1cm} (12)

where \( f(x) \), \( g(x) \) and \( h(x) \) are sufficiently smooth in a domain \( D \subset \mathbb{R}^{n} \) (the mapping \( f: D \to \mathbb{R}^{n}, g: D \to \mathbb{R}^{n} \) are vector fields on \( D \)) and \( x = [x_1 x_2 \ldots x_n]^T \) is a state vector. It is necessary to find a state feedback control \( u \), that transforms the nonlinear system into an equivalent linear system. Clearly, generalization of this idea is not possible in every nonlinear system: there must be a certain structural property that allows performing in such a manner of cancellation.

Using feedback to cancel nonlinearities requires the nonlinear state equation to have a structure:

Definition [12]:

\[ \dot{x} = Ax + By(x)[u - \alpha(x)] \] \hspace{1cm} (13)

where \( A \) is \( n \times n \) and \( B \) is \( n \times p \) matrix, the functions \( \alpha: \mathbb{R}^{n} \to \mathbb{R}^{n}, \gamma: \mathbb{R}^{n} \to \mathbb{R}^{p \times p} \) are defined on domain \( D \subset \mathbb{R}^{n} \) that contains the origin. Furthermore, two conditions must be satisfied. The first one is that the pair \((A, B)\) must be controllable. The second one is that \( \gamma(x) \) must be nonsingular for all \( x \in D \). This is consequence of the control law form: \( u = \frac{1}{\gamma(x)} \) that provides a new control signal \( v \).

Even if the state equation does not have the structure (13), sometimes it is possible to execute feedback linearization for another choice of variables. Therefore, a more comprehensive definition is given [12]:

A nonlinear system:

\[ \dot{x} = f(x) + G(x) \quad u \] \hspace{1cm} (14)

where \( f: D \to \mathbb{R}^{n} \) and \( G: D \to \mathbb{R}^{n \times p} \) are sufficiently smooth on a domain \( D \subset \mathbb{R}^{n} \), is said to be feedback linearizable (or input – state linearizable) if there exist a diffeomorphism \( T: D \to \mathbb{R}^{n} \) such that \( D_Z = T(D) \) contains the origin and the change of variables \( z = T(x) \) transforms the system (14) into the form:

\[ \dot{z} = Az + By(x)[u - \alpha(x)] \] \hspace{1cm} (15)

with \((A,B)\) controllable and \( \gamma(x) \) nonsingular for all \( x \in D \).

V. DETERMINATION OF RELATIVE DEGREE

The relative degree of a linear system is defined as the difference between the poles (degree of the transfer function's denominator polynomial) and zeros (degree of its numerator polynomial). To extend this concept to nonlinear systems more mathematical treatment will be needed. The following definition is given and repeated here for completeness:

Definition [13]: Given the Single Input – Single Output System, SISO, outlined in (12), it is said to have relative degree \( r \) at a point \( x_0 \) if:

i) \( L_g L_f^{1-k} h(x) = 0 \) for all \( x \) in a neighborhood of \( x_0 \) and all \( k < r - 1 \)

ii) \( L_g L_f^{r-1} h(x) \neq 0 \)

The terms \( L_g \) and \( L_f \) represent the Lie derivative of \( h(x) \) taken along \( g(x) \) and \( k \times \text{times} \) along \( f(x) \), respectively.

VI. NONLINEAR MATHEMATICAL MODEL

The nonlinear mathematical model of the DC motor was obtained considering the speed dependent friction nonlinearity. Many models of friction, widely studied in the literature, differ mainly in the description of the moment of friction. These models, generally, describe the friction torque as a static and/or dynamic function of angular velocity [14]. Here, as well as in [14], Tustin friction model was adopted as follows: \( T_{\text{frict}} = T_{\text{stribeck}} + T_{\text{viscous}} = T_s \, \text{sgn}(\dot{\theta}) + (T_s - T_c) e^{-\frac{\dot{\theta}}{\omega_s}} \, \text{sgn}(\dot{\theta}) + B \, \dot{\theta} \).

where \( B \) is the viscous friction coefficient and \( \dot{\theta} \) is Stribeck velocity. It includes viscous friction part \( T_{\text{viscous}} \) and Stribeck function \( T_{\text{stribeck}} \) that is a decreasing function in relation to the velocity increase and with upper bound equal to the static friction torque \( T_s \), at zero velocity, and lower bound equal to
the Coulomb friction torque $T_c$. In this approach, the constant portion of the Coulomb model is replaced by Stribeck function. The viscous component of friction torque is a linear function, and friction curve of Stribeck model is nonlinear function, and they will be considered separately. Therefore, the nonlinear mathematical model of the DC motor is adopted as follows:

$$J_{eq}\dot{\theta}_t + T_{st}(\dot{\theta}_t) + B_{eq,v}\dot{\theta}_t = A_mV_m \quad (16)$$

### Table I

**The Numerical Values of the Plant Parameters**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values and units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_{eq}$</td>
<td>0.0021 kgm$^2$</td>
</tr>
<tr>
<td>$R_m$</td>
<td>2.6 $\Omega$</td>
</tr>
<tr>
<td>$k_t$</td>
<td>0.0077 Nm/A</td>
</tr>
<tr>
<td>$\eta_m$</td>
<td>0.69</td>
</tr>
<tr>
<td>$\eta_p$</td>
<td>0.9</td>
</tr>
<tr>
<td>$K_g$</td>
<td>70</td>
</tr>
</tbody>
</table>

In order to identify friction for the above described DC motor, two distinct experiments, cited here, were performed in paper [14]. In the first, the control voltage is increased gradually at the rate of 0.05 V/s and at the instant when the motor shaft starts to rotate, control voltage is recorded. Ten measurements were done and by these values averaging, the static friction torque was obtained. From (16), it can be seen that if the velocity is kept constant, the friction torque is proportional to the control signal $V_m$. In the second experiment, a linear PI control algorithm was used to stabilise angular velocity to various constant values, and after transient time, the average values of the control voltage and the angular velocity are calculated and recorded. The part of the obtained friction curve $T_{st}(\dot{\theta}_t)$, for low angular velocity values, where the Stribeck effect is dominant, is shown in Fig. 5. It is assumed that friction characteristics are symmetrical, for negative and positive values of angular velocity. Applying standard optimization techniques with Matlab, the friction parameters were obtained, as follows:

$$T_{st} = 0.0174 sgn(\dot{\theta}_t) + 0.0087 e^{-0.064 sgn(\dot{\theta}_t)}, \quad B_{eq,v} = 0.0721 \quad (17)$$

In order to overcome the jump discontinuity of the proposed friction model, at $\dot{\theta}_t = 0$, that jump is replaced by a line of finite slope, up to a very small threshold $\varepsilon$, as is shown in Fig. 5 [14]. The slope is bounded by red dashed lines defined by this threshold.

This line of finite slope will be used only for comparison with the hyperbolic tangent function (Fig. 6), because method of feedback linearization requires differentiable functions (as can be seen from the given definitions in the previous section). In this way only Coulomb and viscous friction is modeled and static friction is neglected. Choosing $x_1 = \theta_t, x_2 = \dot{\theta}_t$ as state variables, $y = \theta_t$ as measured variable and $u = V_m$ as control variable and denoting nonlinearity by $f(x)$, state equation of the system was obtained as follows:

$$\dot{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -B_{eq,v} \\ 0 & J_{eq} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(x) \quad + \begin{bmatrix} 0 \\ A_m \end{bmatrix} u \quad (18)$$

$$y = [1 \quad 0]x \quad (19)$$

To ensure that this model is an equivalent representation of the original system, an experiment was performed, with the results shown below on Fig. 7 for step and Fig. 8 for sinusoidal response.

![Fig. 6. Differential function of the hyperbolic tangent](image1)

![Fig. 7. Experimental results: comparison between real and model data for step input](image2)
VII. EXPERIMENTAL RESULTS

Applying Definition [12] to the system (18) – (19) yields:

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{B_{eq,n}}{J_{eq}} \end{bmatrix} \]  
\[ B = \begin{bmatrix} 0 \\ \frac{A_m}{J_{eq}} \end{bmatrix} \]

\[ \alpha(x) = \frac{J_{eq}}{A_m} f(x) \]  
\[ \gamma(x) = 1. \]

First condition is met:

\[ U = [B AB A^2B \ldots A^{n-1}B]. \]

Order of system is \( n = 2 \) and, because \( \text{rank } U = n \), the pair \((A, B)\) is controllable:

\[ U = [B AB] = \begin{bmatrix} 0 & \frac{A_m}{J_{eq}} \\ \frac{A_m}{J_{eq}} & -\frac{B_{eq,n}}{J_{eq}} \end{bmatrix}. \]

System transformation is not required and all functions are smooth and differentiable. \( \gamma(x) \) is not equal to zero, so the second condition is also met. With both conditions fulfilled feedback linearization is allowed.

The first derivative of the system (18) – (19) output does not depend on the control signal, which means that the relative degree of the system is not 1:

\[ \dot{y} = L_f h(x) + L_g h(x) u. \]

\[ L_g h(x) = 0 \quad \text{and} \quad L_f h(x) = x_2 \]

\[ \dot{y} = \dot{x}_2 = -\frac{B_{eq,n}}{J_{eq}} x_2 - f(x) + \frac{A_m}{J_{eq}} u \]

\[ L_f^2 h(x) = -\frac{B_{eq,n}}{J_{eq}} x_2 - f(x) \]

\[ L_g L_f h(x) = \frac{A_m}{J_{eq}} \]

Conclusion is that relative degree of this system is equal to the system order \( r = 2 \). The desired time-domain specifications for controlling the position of the load shaft are:

overshoot: \( PO = 0\% \) and settling time: \( t_s \leq 2.3 \) s. Choosing the control signal \( u \) in the following form:

\[ u = \frac{1}{L_g L_f h(x)} \left[ -L_f^2 h(x) + v \right] \]

\[ = \frac{1}{J_{eq}} \left[ \frac{B_{eq,n}}{J_{eq}} x_2 + f(x) + v \right] \]

with \( v = -K_o x_1 - K_1 x_2 + K_0 x_{ref} \), where \( K_o = 400 \), \( K_1 = 40 \) were obtained by calculating the minimum damping ratio and natural frequency, which were required to meet the specifications; \( x_{ref} \) is desired output or reference. Linear control is obtained in the same way, with the same coefficients, but without canceling the nonlinearity:

\[ u_1 = -K_o x_1 - K_1 x_2 + K_0 x_{ref} \]

The experiments were performed with Quanser rotary servo motor, SRV02. This model is equipped with the optical encoder and tachometer, for motor position and speed measuring, respectively [14].

\[ \text{Reference} \]
\[ \text{Output controlled by linear controller} \]
\[ \text{Output controlled by nonlinear controller} \]

The experiments were performed with Quanser rotary servo motor, SRV02. This model is equipped with the optical encoder and tachometer, for motor position and speed measuring, respectively [14].
It can be observed, from the Fig. 9, Fig. 10, Fig. 11 and Fig. 12 that the specific requirements are met. The overshoot and the settling time are in the domain of desired values. Furthermore, it is observed that the nonlinear controller is more convenient and has better achievements for position management.

VIII. CONCLUSION

The feedback linearization technique was used for controlling the nonlinear system. The primary aim was to corroborate this method for controlling position of DC motor. First, the modelling of an object has been obtained.

After it has been experimentally confirmed that linear equations did not describe this object well enough, the nonlinear model was presented by including Striebeck model of the friction. Using the concise presentation of the feedback theory the conditions for accomplishing this technique were considered. In order to satisfy those conditions an approximation of the function, which represent nonlinearity, was found as hyperbolic tangent. Then the fulfillment of the conditions for the synthesis of the control law was proven.

At the end it could be observed, through the experiment and analysis results, that the desired response (output signal of the model reference) was tracked by the plant response. The comparison of the linear and nonlinear controller is given. The results show that the controllers, synthesized in this way, are able to satisfy desired position, but that nonlinear controller gives better outcome.

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