Comparison of Several Approaches for Calculating 2D MoM Integrals

Veljko Crnadak and Dragan Olćan

***Abstract*—We present results for several different methods of calculating the singular integral of the Hankel function, that arises when analyzing scattering from an infinitely long conducting strip using method of moments. Particularly we focus on calculation of the diagonal elements in method of moments matrix. We use several different approaches for numerical calculation of the integral: trapezoidal rule, potential integrals evaluation method, double-exponential (DE) method, and finally, Gauss-Legendre (GL) method, with an adequate variable substitution.**

***Index Terms*—convergence; double exponential; Gauss-Legendre; Hankel; integral; method of moments; numerical integration; scattering; singularity; strip.**

I. Introduction

Scattering from infinitely long structures is a well known theoretical problem in computational electromagnetics [1]. Here, we consider a perfectly conducting, infinitely long, infinitely thin strip of a finite width L, placed in a vacuum. It is being illuminated by an incident TM-polarized plane wave, $E^{\mathrm{i}} = \mathrm{e}^{\mathrm{j}k(x\cos\phi^{\mathrm{i}} + y\sin\phi^{\mathrm{i}})}$, as it is shown in Fig. 1, where $k = \frac{2\pi}{\lambda}$ is a wave number. The field that is being radiated by induced surface current of a density $\boldsymbol{J}$, is a scattered field, and it is expressed with a following equation [2],

$$E^{\mathrm{s}} = -\frac{kZ_{\mathrm{c}}}{4}\int_0^L J_{\mathrm{z}}(x')H_0^{(2)}(k|x-x'|)\mathrm{d}x', \quad (1)$$

where $Z_{\mathrm{c}} \approx 120\pi\ \Omega$ is the characteristic impedance of a vacuum, and $H_0^{(2)}(x)$ is the Hankel function of the second kind, zeroth order. On the surface of the strip, the boundary condition for the electric field is, $\boldsymbol{n} \times (\boldsymbol{E}^{\mathrm{i}} + \boldsymbol{E}^{\mathrm{s}}) = 0$, that is $E^{\mathrm{i}} + E^{\mathrm{s}} = 0$, from which it follows,

$$E^{\mathrm{i}} = \frac{kZ_{\mathrm{c}}}{4}\int_0^L J_{\mathrm{z}}(x')H_0^{(2)}(k|x-x'|)\mathrm{d}x'. \quad (2)$$

This is the integral equation with the unknown current distribution J_{z}, that is to be solved numerically.

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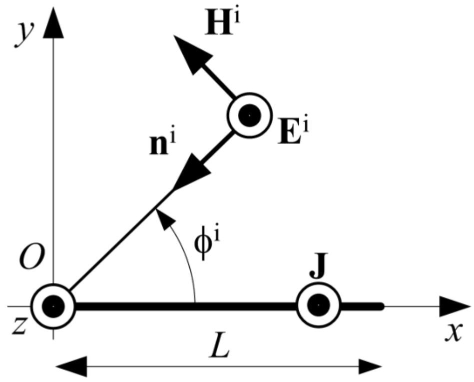

Fig. 1. Coordinate system is bound, to an infinitely long conducting strip of a finite width *L*.

II. Method

In order to solve the integral equation (2), we subdivide the strip, along the x-axis, into N segments of equal width Δx, $\Delta x = \frac{L}{N}$. The incident field is $E^{\mathrm{i}} = \mathrm{e}^{\mathrm{j}kx\cos\phi^{\mathrm{i}}}$. We approximate the unknown density $J_z(x')$, with the finite sum $\sum_{n=1}^{N} a_n f_n$, where a_n are the unknown coefficients, which have to be determined, and f_n are the pulse basis functions, defined in a following way [3],

$$f_n = \begin{cases} 0, \text{for } x' < (n-1)\Delta x \\ 1, \text{for } (n-1)\Delta x \le x' \le n\Delta x. \\ 0, \text{for } x' > n\Delta x \end{cases} \quad (3)$$

The applied approximation is known as a staircase (piecewise-constant) approximation. For the test function we use Dirac delta function, positioned at the midpoint of the *m*-th segment, $w_m = \delta(x_m)$, where x_m is the point at the middle of the *m*-th segment, $x_m = (m-0.5)\Delta x$, $m = 1, 2, \ldots, N$. In (2), the unknown density is substituted with the mentioned finite sum, for $0 \le x \le L$ we have,

$$\mathrm{e}^{\mathrm{j}kx\cos\phi^{\mathrm{i}}} = \frac{kZ_{\mathrm{c}}}{4}\sum_{n=1}^{N} a_n \int_{(n-1)\Delta x}^{n\Delta x} H_0^{(2)}(k|x-x'|)\mathrm{d}x'. \quad (4)$$

If the inner product of the test function and the left side of the equation (4) is equalized, with the inner product of the test function and the right side of the equation (4), then the system of linear equations is obtained, for $m = 1, 2, \ldots, N$,

$$\mathrm{e}^{\mathrm{j}kx_m\cos\phi^{\mathrm{i}}} = \frac{kZ_{\mathrm{c}}}{4}\sum_{n=1}^{N} a_n \int_{(n-1)\Delta x}^{n\Delta x} H_0^{(2)}(k|x_m-x'|)\mathrm{d}x'. \quad (5)$$

The solution of the system (5), is a vector $(a_n)_{N\times1}$, expressed in a following way,

$$(a_n)_{N\times1} = \begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix}^{-1} \begin{pmatrix} e^{jkx_1\cos\phi^i} \\ \vdots \\ e^{jkx_N\cos\phi^i} \end{pmatrix}. \quad (6)$$

Element z_{mn} of the matrix, $\begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix}$, for $m\neq n$ is,

$$z_{mn} = \frac{kZ_c}{4}\int_{(n-1)\Delta x}^{n\Delta x} H_0^{(2)}(k|x_m - x'|)\mathrm{d}x'. \quad (7)$$

The value of the diagonal element z_{nn} is,

$$z_{nn} = \frac{kZ_c}{4}\int_0^{\Delta x} H_0^{(2)}(k|0.5\Delta x - x'|)\mathrm{d}x'. \quad (8)$$

The Hankel function, $H_0^{(2)}(k|x|)$, has a singularity at zero, as it is shown in Fig. 2. This means that the integrand in (8), has a singularity at $0.5\Delta x$. We use the small argument approximation of the Hankel function [2],

$$H_0^{(2)}(k|x|) \approx 1 - \frac{j2}{\pi}\ln\frac{\gamma k|x|}{2}, x \to 0, \quad (9)$$

where $\gamma \approx 1.781$ is Euler's constant. If in (8), we apply the small argument approximation (9), then we have a new expression for the value of the diagonal element [2],

$$z_{nn} \approx \frac{kZ_c}{4}\int_0^{\Delta x}\left[1 - \frac{j2}{\pi}\ln\left(\frac{\gamma k|0.5\Delta x - x'|}{2}\right)\right]\mathrm{d}x'. \quad (10)$$

Because integrands in (8) and (10), have a singularity at $0.5\Delta x$, integrals in (8) and (10) are hard to calculate numerically. In [1], there is a well-known approximation of the value of the diagonal element (10),

$$z_{nn} = \frac{kZ_c\Delta x}{4}\left[1 - \frac{j2}{\pi}\ln\left(\frac{\gamma k\Delta x}{4e}\right)\right], \quad (11)$$

where e is the base of the natural logarithm. One of the alternative methods, for the determination of the value of the diagonal element, is trapezoidal rule, where the interval from 0 to Δx, is subdivided into N_a equal subintervals, $\Delta x_a = \frac{\Delta x}{N_a}$, so we have,

$$z_{nn} \approx \frac{kZ_c\Delta x_a}{4}\sum_{i=1}^{N_a} H_0^{(2)}(k|0.5\Delta x - (i-0.5)\Delta x_a|). \quad (12)$$

This method, for $N_a = 10^8$, $N = 300$, $f = 300$ MHz and $L = 3\lambda$, gives the value of the diagonal element,

$$z_{nn} = 5.92127561322935 + j17.2509930095781\ \Omega, \quad (13)$$

while equation (11), for the same values of N, f and L, gives the value of the diagonal element,

$$z_{nn} = 5.92176264065361 + j17.2522776982593\ \Omega. \quad (14)$$

Before the application of numerical methods on solving the integral in (8), let us map the interval of integration in (8), into interval $[-1,1]$, $[0,\Delta x] \to [-1,1]$. We begin with the general interval of integration $[l_1, l_2]$, and let us map it onto interval $[-1,1]$, using the system of linear equations,

$$\begin{cases} -1 = Al_1 + B \\ \ \ 1 = Al_2 + B \end{cases}. \quad (15)$$

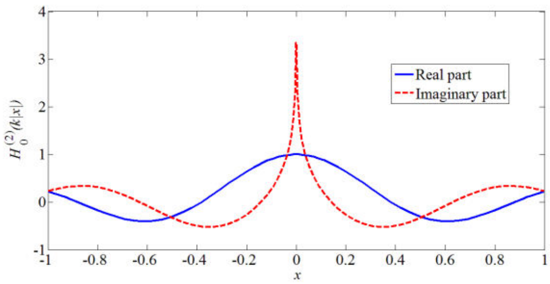

Fig. 2. Diagram of the Hankel function of the second kind and zeroth order.

By solving the system (15) we get, $A = \frac{2}{l_2 - l_1}$ and $B = -\frac{l_2 + l_1}{l_2 - l_1}$. For the interval of integration in (8), and also for already mentioned values for N, f and L (for which we have obtained the results), we have, $A = 200$ and $B = -1$. If in (8), variable x' is substituted by variable y, using the equation, $x' = \frac{y+1}{200}$, and the symmetry of the function $H_0^{(2)}(\frac{k}{200}|y|)$, then we have,

$$z_{nn} = \frac{kZ_c}{400}\int_0^1 H_0^{(2)}\left(\frac{k}{200}y\right)\mathrm{d}y. \quad (16)$$

Let us first consider the numerical method efficient in solving the integrals that have singularity at one or both endpoints of the integration interval, which is known as double exponential (DE) method [4]. Integral in (16) can be solved by applying the DE method. Variable y is substituted by variable t, using the equation [4],

$$y = \phi(t) = \frac{1}{2}\left[\tanh\left(\frac{\pi}{2}\sinh t\right) + 1\right], \quad (17)$$

that transforms the expression (16) into a following form,

$$z_{nn} = \frac{kZ_c}{400}\int_{-\infty}^{+\infty} H_0^{(2)}\left(\frac{k}{200}\phi(t)\right)\frac{\frac{\pi}{4}\cosh t}{(\cosh(\frac{\pi}{2}\sinh t))^2}\mathrm{d}t. \quad (18)$$

If we apply the trapezoidal rule on integral in (18), we get,

$$z_{nn} \approx \frac{kZ_c h}{400}\sum_{i=-N}^{N} H_0^{(2)}\left(\frac{k}{200}\phi(ih)\right)\frac{\frac{\pi}{4}\cosh ih}{(\cosh(\frac{\pi}{2}\sinh ih))^2}, \quad (19)$$

where h is the mesh size, $h = 2^{-M}$, and $N = 6\cdot 2^M$. Integral in (18), can be replaced with the finite sum of $2N+1$ elements, in (19), due to rapidly decreasing weight coefficient,

$$\phi'(t) = \frac{\frac{\pi}{4}\cosh t}{(\cosh(\frac{\pi}{2}\sinh t))^2} \approx O\left(e^{-\frac{\pi}{2}e^{|t|}}\right), |t| \to \infty. \quad (20)$$

The name of the method, double exponential, originates from the characteristic of the weight coefficient (20). Criterion, that is used to determine whether an array, $\{z_{nn}\}$, has converged or not, is the following. We observe the absolute value of the difference between successive elements of an array,

$$\Delta z_{nn}(N) = |z_{nn}(N+1) - z_{nn}(N)|. \quad (21)$$

If $\Delta z_{nn}(N)$, after monotonous decreasing, has entered into rapid changes, as shown in Fig. 3, then it is assumed that an array $\{z_{nn}\}$ has converged.

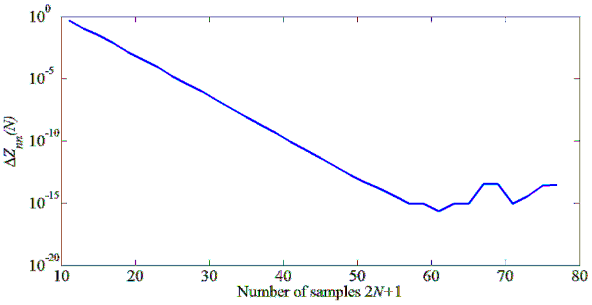

Fig. 3. Absolute value of the difference between successive elements of the DE array $\{z_{nn}\}$.

The DE array $\{z_{nn}\}$ converges to the value,

$$z_{nn} = 5.92127561322414 + \mathrm{j}17.2509930357105\ \Omega. \quad (22)$$

As the next approach for solving the integral (8), we apply the Gauss-Legendre method (GL). Corresponding array $\{z_{nn}\}$ will not converge, as it is shown in Fig. 4.

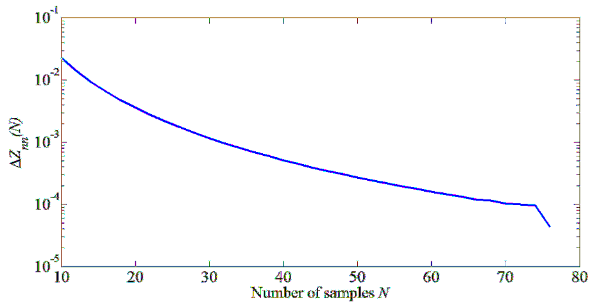

Fig. 4. Absolute value of the difference between successive elements of the GL array $\{z_{nn}\}$.

In the case of array $\{z_{nn}\}$, obtained by the GL method, there is no convergence, because it is not possible to approach the singularity near enough, which is different from the DE method, where such a thing is possible.

Let us replace in (16), variable y with variable t, in the following way,

$$t = \frac{\ln(1+py)}{\ln(1+p)}, \text{ where } p \text{ is a parameter.} \quad (23)$$

Substitution (23) transforms (16) into the equation,

$$z_{nn} = kZ_{\mathrm{c}} \frac{\ln(1+p)}{400p} \int_0^1 H_0^{(2)} \left\{ \frac{k[\mathrm{e}^{t\ln(1+p)} - 1]}{200p} \right\} \mathrm{e}^{t\ln(1+p)} \mathrm{d}t. \quad (24)$$

Let us denote the integrand in (24), with $f(t)$. For the function $f(t)$, when $t \to 0^+$, we have,

$$f(t) \approx \left[1 - \frac{\mathrm{j}2}{\pi} \ln \frac{\gamma k t \ln(1+p)}{400p}\right] [1 + t\ln(1+p)]. \quad (25)$$

From (25) we see, that the factor $t\frac{\ln(1+p)}{400p}$, controlled by the value of the parameter p, enables us to approach the singularity, as close as we want. Large values of the $f(t)$, in the proximity of the singularity, are attenuated by the factor $\frac{\ln(1+p)}{400p}$, that stands in front of the integral in (24). If we apply the GL method for solving the integral (24), the corresponding array $\{z_{nn}\}$ converges to the value in (22), for $p = 10^{12.1}$. Let us replace in (16), variable y with variable t, in the following way,

$$t = \left[\frac{\ln(1+py)}{\ln(1+p)}\right]^{\frac{1}{l}}, \quad (26)$$

where p and l are parameters, $l = 2, 3, 4$.

Substitution (26) transforms (16), into the equation,

$$z_{nn} = kZ_{\mathrm{c}} \frac{\ln(1+p)^l}{400p} I, \text{ where } I \text{ is,} \quad (27)$$

$$I = \int_0^1 H_0^{(2)} \left\{ \frac{k}{200p} \left[\mathrm{e}^{t^l \ln(1+p)} - 1\right] \right\} \mathrm{e}^{t^l \ln(1+p)} t^{l-1} \mathrm{d}t. \quad (28)$$

Let us denote the integrand in (28), with $g(t)$. For the function $g(t)$, when $t \to 0^+$, we have,

$$g(t) \approx \left[1 - \frac{\mathrm{j}2}{\pi} \ln \frac{\gamma k t^l \ln(1+p)}{400p}\right] [1 + t^l \ln(1+p)] t^{l-1}. \quad (29)$$

From (29) we see, that the factor $\frac{t^l \ln(1+p)}{400p}$, controlled by the values of the parameters p and l, enables us to approach the singularity, as close as we want. Large values of the Hankel function, in the proximity of the singularity, are attenuated by the factor $\frac{\ln(1+p)^l}{400p} t^{l-1}$. If we apply the GL method for solving the integral (28), the corresponding array $\{z_{nn}\}$ converges, for $p = 10^9$, if $l = 2$, for $p = 10^{4.9}$, if $l = 3$, and for $p = 10^3$, if $l = 4$, which is found by numerical experiments.

III. Results

Integrand f, expression (24), multiplied by the attenuation factor $\frac{\ln(1+p)}{400p}$, for $p = 10^{12.1}$, is shown in Fig. 5. Integrand g, expression (28), multiplied by the attenuation factor $\frac{\ln(1+p)^l}{400p}$, for $p = 10^{4.9}$ and $l = 3$, is shown in Fig. 6. Substitutions of variable y with variable t, given in (23) and (26), have eliminated the influence of the singularity of the Hankel function, which is shown in Figs. 5 and 6. Relative error of the element z_{nn}, evaluated using (27) and compared to the corresponding reference values, for four different pairs of values p and l, is shown in Figs. 7, 8 and 9. On all three pictures (Figs. 7, 8 and 9), relative error of the element z_{nn}, evaluated using (8) and compared to the corresponding reference values, also appears. Values of the elements z_{nn}, given in (8) and (27), are calculated by the GL method.

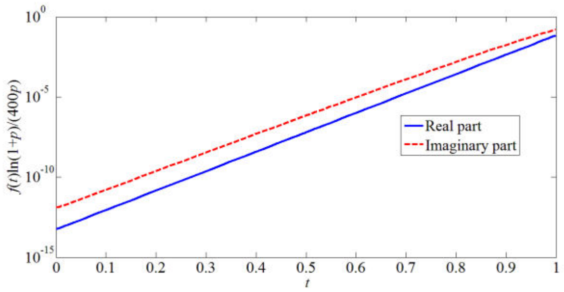

Fig. 5. Integrand f multiplied by the attenuation factor.

Family of curves, in the Fig. 7, was obtained for the reference value (13), in the Fig. 8, for the reference value (14), and in the Fig. 9, for the reference value (22). In the Fig. 10 it can be seen that the GL array $\{z_{nn}\}$, given in (27), has converged for the corresponding pairs of values p and l, while the GL array $\{z_{nn}\}$, given in (8), has not converged.

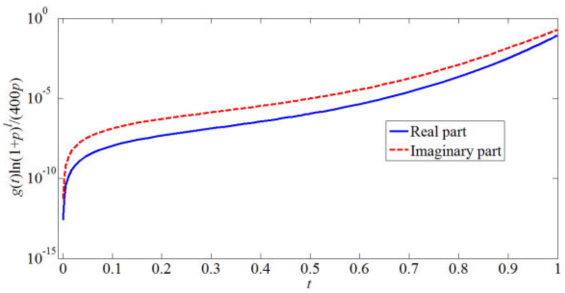

Fig. 6. Integrand g multiplied by the attenuation factor.

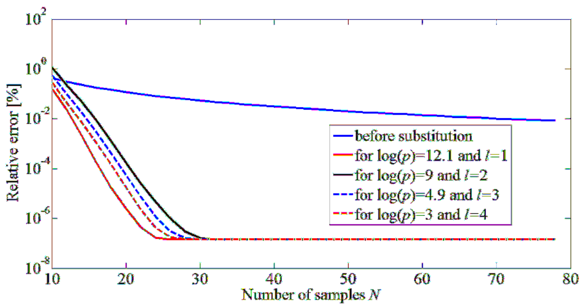

Fig. 7. Relative error of the element z_{nn}, compared to the reference value.

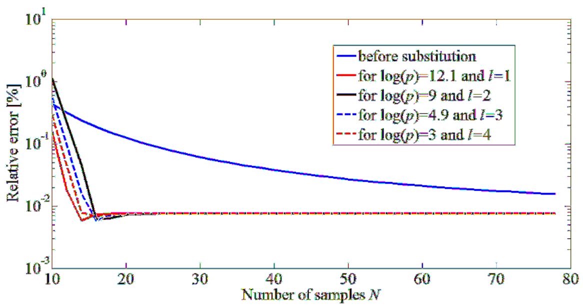

Fig. 8. Relative error of the element z_{nn}, compared to the reference value.

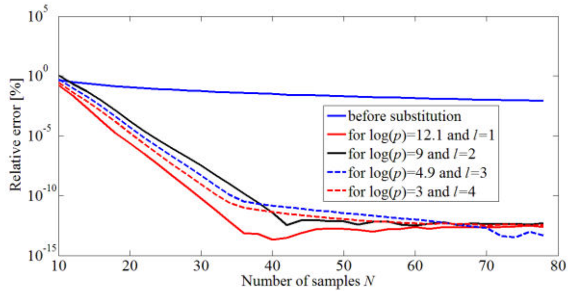

Fig. 9. Relative error of the element z_{nn}, compared to the reference value.

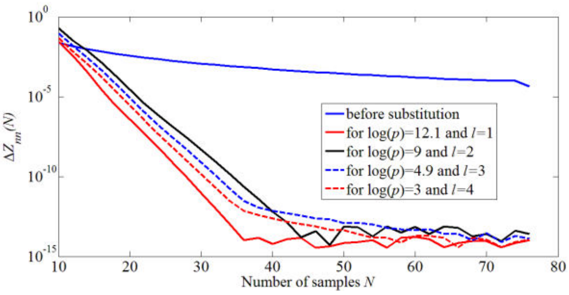

Fig. 10. Absolute value of the difference between successive elements of the GL array $\{z_{nn}\}$.

IV. Conclusion

In this paper we have demonstrated a solution for the problem of calculating the values of the diagonal elements, when analyzing 2D scatterers using method of moments. The cause of the problem is the singularity of the Hankel function. It has been shown, that by applying the GL method, with the adequate variable substitution ($p = 10^{12.1}$ and $l = 1$), the same value of the diagonal element is obtained (22), as with the DE method. The GL method, with the adequate variable substitution ($p = 10^{12.1}$ and $l = 1$), converges faster towards the value (22), compared to the double exponential method. The drawback of the GL method, with the adequate variable substitution ($p = 10^{12.1}$ and $l = 1$), compared to the DE method, lays in its limited application as the proposed substitution can not be applied to other functions.

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