INTERACTION REPRESENTED TRANSPORT EQUATION IN HALF SPACE

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Abstract - Using analytical form of semigroups generated by linear scattering Boltzmann operator a transport equation valid for half space problems is being obtained.

1 INTRODUCTION

The subject of this paper is the linear time dependent transport equation of one speed particles in plane geometry

\[
\frac{\partial}{\partial t} \Psi_{\gamma}(x,\mu,\lambda) = -W_{\gamma}(x,\mu,\lambda) + \sum_{\gamma'} \int_{\Sigma} \frac{\partial}{\partial t} \Psi_{\gamma'}(x',\mu',\lambda') d\lambda' + Q(x,\mu,\lambda)
\]

expressed as a formal Cauchy problem:

\[
\psi_t - H\psi + Q,
\]

\[
\psi_0 = \psi(x,\mu,\lambda = 0)
\]

where the linear transport operator \(H\) is \(H = \mu B + A\).
Here the linear collision transport operator \(B\) for simplicity is supposed for being isotropic in units

\[
\Sigma = \frac{1}{\mu}, \quad \lambda = \frac{1}{\beta}, \quad \mu = 1 - \mu_p c = \frac{\Sigma}{\lambda}. \quad \text{The projector} \quad P = \int_0^\infty \psi(x,\mu,\lambda) d\lambda
\]

comes as result of isotropic scattering. Using analytical expressions of semigroups \(U(t) = \exp(\pm tB)\):

\[
e^{\pm tB} = e^{\pm t} \left( I \mp \frac{(1 - e^{\pm tB})}{\xi} B \right)
\]

an equivalent transport equation in interaction form is obtained [1-2]

\[
\frac{\partial}{\partial t} \phi - M(\mu) \phi + S,
\]

\[
\phi_0 = \phi(x,\mu,\lambda = 0)
\]

where

\[
\phi = U_\Psi\psi, \quad S = U_\Psi Q
\]

and

\[
M(t) = M + 2\pi \delta(\xi, \lambda^2 MP - e^t MP),
\]

\[
\xi = c\ell 2, \quad MP(\mu) = \mu P(\mu)
\]

later on this model has been successfully extended to three dimensional geometry and analytically solved also [3-6]. General solution of Eq(4) is expressed through shifted coordinate

\[
\phi(q, t) = S(q, t) + \Psi(q, \lambda)(\lambda(t)),
\]

\[
q = (x, \mu), \quad \lambda(t) = \lambda(t)^2 = \lambda(t)^2
\]

where

\[
\lambda(t) = M + a(t) MP + a(-t) PM
\]

Obviously in case of half space, e.g. \(x > 0\) the shifted coordinate \(x'\) may violate the positivity condition and therefore Eq(7) need not be a solution. This problem will be considered in the sequel.

2. SOLUTION OF THE EQUATION IN HALF SPACE

In order to simplify the analysis we consider homogeneous part of Eq(4). Also we observe that the shifted coordinate come due to action of Abelian shift group \(e^{-\beta(t)}\) appearing in operational solution

\[
e^{-\beta(t)} f(x) = f(x - tH)\beta(t)
\]

In case of half space the Abelian group \(e^{-\beta(t)}\) becomes a semigroup. The question is now how to relate solution of Eq(4) and this semigroup due to boundary condition at \(x = 0\). Interesting is that the boundary conditions may be formulated either as an improvement of infinite media problem or as a semigroup model in half space.

The main result can be formulated in

Theorem. The solution (7) is valid for half space too.

Proof. To prove this we shall use the Laplace transformation to Eq(4) with respect to space.
coordinate i.e. $F(x, \mu, t) = \int_0^1 e^{-(\lambda-\mu)x} \Phi(\mu, x, \lambda) d\lambda$.

Keeping in mind the transformation of differentiation operator we get

$$\hat{F}(\mu, t) = e^{-\nu t} \int_0^1 e^{-\nu \lambda} \Phi(\mu, x, \lambda) d\lambda$$

whence solution is

$$F(x, \mu, t) = e^{-\nu t} \int_0^1 e^{-\nu \lambda} \Phi(\mu, x, \lambda) d\lambda$$

Applying here the inverse Laplace transformation we find

$$\Phi(x, \mu, t) = \mathcal{L}^{-1} \{e^{-\nu \lambda} \Phi(\mu, x, \lambda) \}$$

There has been proved [2] that Eq.(4) can be solved successfully using an approximate $\Lambda(t)$ defined by some point $0 < t < T$. In this case [12] reduces to

$$\Phi(x, \mu, t) = \mathcal{L}^{-1} \{e^{-\nu \lambda} \Phi(\mu, x, \lambda) \}$$

and finally

$$\Phi(x, \mu, t) = \mathcal{L}^{-1} \{e^{-\nu \lambda} \Phi(\mu, x, \lambda) \}$$

Since the continuous spectrum of $\Lambda(t)$ comprises the interval $[-1, 1]$ and due to the causality condition $x < 0$ we find $x - t\Lambda^2 < t(1 - \Lambda^2)$ < 0. Thus the last term in this equation vanishes as nonphysical one. Therefore, we have got the same result as in an infinite medium case.

As a second approach to prove the theorem we shall use the shift operator method which is at the same time an explanation of preceding result as well as an insight to boundary conditions modeling. For this we define two shift operators

$$x(x, x, \mu, t) = x(x, \mu, t), \quad \Phi(x, x, \mu, t) = \Phi(x, \mu, t)$$

The Eq.(14) will be written for some $\lambda \geq 0$ as

$$(\lambda + K)\Phi(x, \mu, t) = \Phi(x, \mu, t)$$

where $K = \frac{1}{\Lambda^2} (1 - \Lambda^2) \Delta(x)$. Observing that

$$K\Phi(x, \mu, t) = \Phi(x, \mu, t)$$

the analytical solution to this equation is

$$\Phi(x, \mu, t) = \sum_{\lambda} (\lambda + K)^{\lambda} \Phi(x, \mu, t)$$

That is we have got the infinite medium solution. The term appearing here is a normalization factor of semi-infinite medium.

**DISCUSSION**

Some care must be taken into account, when evaluating the expression (7) due to boundary conditions at infinity i.e. Thus, since $\Phi$ must be decreasing function for $x > 0$, and $\Phi$ is positive, only $0 < \mu$ spectral term in Eq.(7) is allowed. With the same reasoning for $x < 0$ we conclude that only negative spectral terms will appear in this case. Practically that means, for some $x < 0$ the actual space coordinates $x' = x - vf$ or $x' = x - (1 - v)f$ may take negative values, for example for discrete eigenvalues. In such a case, the distribution for negative space coordinate is being used instead. A similar consideration for negative space coordinates applies. Interestingly to mention here that in a half space case, the translation group property is violated. However, the left boundary flux distribution will appear when applying the exponential function of differentiation operator. Consequently, since the Fourier transform, in this case Fourier transform induced by Boltzmann transport operator $B$, is an isomorphism, the vacuum boundary condition is automatically included.

**REFERENCES**


**INTERAKCIONO PRIKAZANA TRANSPORTNA JEDNAČINA U POLUPROSTORU**

V. Stančić

Korišćenjem analitičkih oblika polugrupa generisanih Boltzmannovim transportnim operatorom dobijeni je analitičko rešenje za poluprostor.